NEWS
OF THENATIONAL ACADEMY OF SCIENCES OF THE REPUBLIC OF KAZAKHSTAN
PHYSICO-MATHEMATICAL SERIES
ISSN 1991-346X **https://doi.org/10.32014/2020.2518-1726.4**
Volume 1, Number 329 (2020), 32 – 37

UDC 517.927

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ON BASIS PROPERTY OF SYSTEMS ROOT VECTORS OF A LOADED MULTIPLE DIFFERENTIATION OPERATOR

Abstract. In the case of non-self-adjoint ordinary differential operators, the basis property of systems of eigenfunctions and associated functions (E&AF), in addition to the boundary value conditions, can be affected by values of coefficients of the differential operator. Moreover, it is known that the basic properties of E&AF can be changed at a small change of values of the coefficients. This fact was first noted in V.A. Il'in. Ideas of V.A. Il'in for the case of non-self-adjoint perturbations of the self-adjoint periodic problem were developed in A.S. Makin where operator was changed due to perturbation of one of the boundary value conditions.

In Sadybekov M.A., Imanbaev N.S., we studied another version of the non-self-adjoint perturbation of the self-adjoint periodic problem. In contrast to A.S. Makin, in Sadybekov M.A. and Imanbaev N.S. perturbation occurs due to the change in the equation, which belongs to the class of so-called loaded differential equations, where the basic properties of root functions are investigated.

In this paper we consider perturbations of a second order differential equation of the spectral problem with a loaded term, containing a value of the unknown function at the point zero, with regular, but not strongly regular boundary value conditions. Question about basis property of eigenfunctions and associated functions (E&AF) systems of a loaded multiple differentiation operator is studied.

Keywords: Eigenvalues, eigenfunctions, associated functions, adjoint operator, multiple differentiation, loaded operaor, Riesz basis, root vectors.

Mathematics subject classification: 34B09, 34L15, 34L05

1. Introduction

In the case of non-self-adjoint ordinary differential operators, the basis property of systems of eigenfunctions and associated functions (E&AF), in addition to the boundary value conditions, can be affected by values of coefficients ofthe differential operator. Moreover, it is known that the basic properties of E&AF can be changedat a small change of values of the coefficients. This fact was first noted in [1]. Ideas of [1] for the case of non-self-adjoint perturbations of the self-adjoint periodic problem were developed in [2], [3], where operator was changed due to perturbation of one of the boundary value conditions.

In [4], we studied another version of the non-self-adjoint perturbation of the self-adjoint periodic problem. In contrast to [2], [3], in [4] perturbation occurs due to the change in the equation, which belongs to the class of so-called loaded differential equations, where the basic properties of root functions are investigated. Problems about the basis properties of root functions of loaded differential operators are thoroughly studied in [5], [6]. In these papers, it was possible to extend the spectral decomposition method of V.A.Il'in [1] to the case of loaded differential operators. By the other method questions about the basis property of functional-differential equations were investigated in [7].

The Riesz basis property of the E&AFsystem of periodic and anti-periodic Sturm-Liouville problems was studied in [8].

In [9], consisting of the Sturm-Liouville equation, together with eigenparameter that depended on boundary conditions and two supplementary transmission conditions; we constructed the resolvent operator and prove theorems on expansions in terms of eigenfunctions in modified Hilbert Space $L_2(a,b)$.

The basis properties in $Lp(-1,1)$ of root functions of the nonlocal problems for the equations with involution have been studied in [10]. Moreover, using these asymptotic formulas, we proved that the root functions of these operators form a Riesz basis in the space $L_2(0,1)$ [11].

In the case when the potential is zero, the system of eigenfunctions of the periodic problem is usual trigonometric system, which forms a complete orthonormal system in $L_2(0,1)$. And if the potential is non-zero, then additional research is required, which answer is the results of [4].

2. Problem Statement and Main Result

Inthispaperweconsiderperturbations of equation of the following spectral problem with a loaded term containing value of the unknown function at the point zero:

$$L_1(u) \equiv -u''(x) + \overline{q(x)}u(0) = \lambda u(x), \quad q(x) \in L_2(0,1), \quad 0 < x < 1, \tag{1}$$

$$U_1(u) = u(0) - u(1) = 0, \quad U_2(u) = u'(1) = 0 \tag{2}$$

Firstweconstruct characteristic determinant of the spectral problem

(1)-(2). Assumingthat $u(0)$ isasomeindependentconstant, it is easy to prove that,when $\lambda \neq 0$, general solution of (1)can be represented as follows:

$$u(x) = C_1 \cdot \cos\sqrt{\lambda}x + C_2 \cdot \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} + u(0)\int_0^x \overline{q(\xi)} \frac{\sin\sqrt{\lambda}(x-\xi)}{\sqrt{\lambda}} d\xi \tag{3}$$

Hence, supposingfirst $x = 0$, andthensatisfying (3) bytheboundaryvaluecondition (2), wegetthe system of three equations, which can be represented in the following vector-matrix form:

$$\begin{bmatrix} -1 & 0 & 1 \\ -\cos\sqrt{\lambda} & -\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} & 1 - \int_0^1 \overline{q(\xi)} \frac{\sin\sqrt{\lambda}(1-\xi)}{\sqrt{\lambda}} d\xi \\ \sqrt{\lambda}\sin\sqrt{\lambda} & -\cos\sqrt{\lambda} & -\int_0^1 \overline{q(\xi)} \cos\sqrt{\lambda}(1-\xi) d\xi \end{bmatrix} \cdot \begin{bmatrix} C_1 \\ C_2 \\ u(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Byusing simple calculations, we obtain

$$\Delta_1(\lambda) = -1 \cdot \left(\frac{1}{\sqrt{\lambda}} \int_0^1 \overline{q(\xi)} \cos\sqrt{\lambda}(1-\xi) \sin\sqrt{\lambda} d\xi + \cos\sqrt{\lambda} - \frac{1}{\sqrt{\lambda}} \int_0^1 \overline{q(\xi)} \sin\sqrt{\lambda}(1-\xi) \cos\sqrt{\lambda} d\xi \right) +$$
$$+ 0 + \left(\cos^2\sqrt{\lambda} + \sin^2\sqrt{\lambda} \right) \tag{4}$$

After the standard transformation of (4), we find that the characteristic determinant $\Delta(\lambda)$ of the spectral problem (1) - (2) is represented as

$$\Delta_1(\lambda) = 1 - \cos\sqrt{\lambda} - \frac{1}{\sqrt{\lambda}} \int_0^1 \overline{q(\xi)} \sin\sqrt{\lambda}\xi d\xi \tag{5}$$

Now we define the operator L_1^*. Using the Lagrange formula for all functions $u \in D(L_1)$ and $v \in D(L_1^*)$, due to boundary value conditions (2), we find:

$$\int_0^1 L_1(u)\overline{v(x)}dx - \int_0^1 u(x)\overline{L_1^*(v)}dx = \overline{v(0)} \cdot u'(0) + u(0) \cdot \left(\overline{v'(1)} - \overline{v'(0)} + \int_0^1 q(x)\overline{v(x)}dx \right) - \int_0^1 \overline{v''(x)}u(x)dx$$

Consequently, theoperator L_1^* isaconjugateoperator to the operator L_1, which is given by the differential expression:

$$L_1^*(v) \equiv -v''(x) = \overline{\lambda}v(x), \quad 0 < x < 1 \tag{1a}$$

and the boundary-value conditions

$$V_1(v)=v'(0)-v'(1)=\int_0^1 q(x)v(x)dx, \quad q(x)\in L_2(0,1), \quad V_2(v)=v(0)=0\,. \qquad (2b)$$

If $q(x)\equiv 0$, then this problem is called Samarskii - Ionkin problem [13]. Inthiscase, boundaryvalueconditions (2) and (2b) are regular, but not strongly regular boundary value conditions [13]. Characteristic determinant of the Samarskii-Ionkin problem will be $\Delta_0(\lambda)=\sqrt{\lambda}\left(1-\cos\sqrt{\lambda}\right)$. The number $\lambda_0^0=0$ isasimpleroot, thatis single eigenvalue, and $v_0(x)=\sqrt{3}x$ is a corresponding eigenfunction of the Samarskii-Ionkin problem. O ther eigen values of the Samarskii-Ionkin problem are double: $\lambda_k^0=(2k\pi)^2, \quad k=1,2,3,\ldots$.

Moreover, $v_{k0}^0=\sqrt{2}\sin(2k\pi x)$ are the ir corresponding eigen functions, and $v_{k1}^0=\frac{\sqrt{2}}{2}x\cos(2k\pi x)$ are associated functions.

Due to the biorthogonality conditions $\left(v_{k1}^0,u_{k1}^0\right)=1$ we have that $u_{k1}^0=4\sqrt{2}\cos(2k\pi x)$ is an eigen function and $u_{k0}^0=2\sqrt{2}(1-x)\sin(2k\pi x)$ is an associated function of the conjugate problem to the Samarskii-Ionkin problem. The system $\left\{u_{k0}^0,u_{k1}^0\right\}$ forms Riesz basis in $L_2(0,1)$ [13].

Function $q(x)$ can bep resented in the form of biorthogonal expansion in a Fourier series by the system $\left\{u_{k0}^0,u_{k1}^0\right\}$:

$$q(x)=\sum_{k=1}^{\infty}a_{k0}u_{k0}^0+\sum_{k=0}^{\infty}a_{k1}u_{k1}^0=\sum_{k=1}^{\infty}a_{k0}\cdot 2\sqrt{2}(1-x)\sin(2k\pi x)+\sum_{k=0}^{\infty}a_{k1}\cdot 4\sqrt{2}\cos(2k\pi x). \qquad (6)$$

Using (6), wefind more convenient representation of the determinant $\Delta_1(\lambda)$. To do it first we calculate the integral in (5). Simple calculations show that

$$\int_0^1\overline{q(\xi)}\sin\sqrt{\lambda}\xi d\xi=2\sqrt{2\lambda}\left(1-\cos\sqrt{\lambda}\right)\left(\sum_{k=0}^{\infty}\left[\frac{2k\pi\overline{a_{k0}}}{\left(\lambda-(2k\pi)^2\right)^2}+\frac{2\overline{a_{k1}}}{\lambda-(2k\pi)^2}\right]\right).$$

By usingthisresult, determinant (5) after conversion is converted to

$$\Delta_1(\lambda)=\Delta_0(\lambda)\cdot A(\lambda), \qquad (7)$$

Where
$$A(\lambda)=1-4\sqrt{2}\sum_{k=1}^{\infty}\left(\pi\overline{a_{k0}}\frac{k}{\left(\lambda-(2k\pi)^2\right)^2}+\overline{a_{k1}}\frac{k}{\lambda-(2k\pi)^2}\right).$$

Therefore, it is proved

Theorem2.1. Characteristic determinant of the spectral problem (1) - (2) when $q(x)\neq 0$ can be represented in the form (7), where $\Delta_0(\lambda)$is the characteristic determinant of the Samarskii - Ionkin problem, a_{k0}, a_{k1} are Fourier coefficients of the biorthogonal expansion (6) of the function $q(x)$ by the E&AF system of the conjugate Samarskii-Ionkin spectral problem.

Thefunction $A(\lambda)$in (7) has poles of the second and first orders at points $\lambda=\lambda_k^0$, but the function $\Delta_0(\lambda)$has zeros of the second order at these points. Thus, thefunction $\Delta_1(\lambda)$, representedbytheformula(7), is an entire analytic function of the variable λ .

Ifatsomeindex j coefficientsof(6) $a_{j0}=0, a_{j1}=0$, then $\lambda_j^1=\lambda_j^0$ is a double eigenvalue of the spectral problem (1) – (2).

If $a_{j0} = 0,\ a_{j1} \neq 0$, then $\lambda_j^1 = \lambda_j^0$ isasimpleeigenvalue of the spectral problem (1) – (2).

The characteristic determinant (7) looks more simply, when $q(x)$ is represented as (6) with a finite first sum. That is, when there exists a number N such that $\overline{a_{k0}} = 0,\ \overline{a_{k1}} = 0$ for all $k > N$. Inthiscase, the (7) -thformulatakesthefollowingform

$$\Delta_1(\lambda) = \Delta_0(\lambda)\left[1 - 4\sqrt{2}\sum_{k=1}^{N}\left(\pi\overline{a_{k0}}\frac{k}{\left[\lambda - (2k\pi)^2\right]^2} + \overline{a_{k1}}\frac{k}{\lambda - (2k\pi)^2}\right)\right] \tag{8}$$

From this particular case of the formula (8) we justify the following

Corollary 2.1. For any pre-assigned numbers: complex $\tilde{\lambda}$ and natural $\tilde{m}$, there always exists a function $q(x)$ such that $\tilde{\lambda}$ is an eigenvalue of the problem (1) - (2) of the multiplicity $\tilde{m}$.

From the analysis of (8), we note, that $\Delta_1\left(\lambda_k^0\right) = 0$ for all $k > N$. That is all eigen values $\lambda_k^0,\ k > N$, of the Samarskii-Ionkin problem are eigenvalues of the spectral problem (1) - (2). Moreover, multiplicity of the eigen values $\lambda_k^0,\ k > N$, is also preserved.

From the orthogonality condition $q(x) \perp v_{j0}^0,\quad q(x) \perp v_{j1}^0$ for all $j > N$ it follows, that in this case

$$\int_0^1 \overline{q(x)} v_{j0}^0(x)dx = \int_0^1 \overline{q(x)} v_{j1}^0(x)dx = 0.$$

Therefore, eigenfunctions $v_{j0}^0(x)$ andassociatedfunctions $v_{j1}^0(x)$ oftheSamarskii-Ionkinproblemforall $j > N$ satisfythespectralproblem(1)–(2) and, consequently, they are eigenfunctions and associated functions of the spectral problem (1) – (2). Thus, inthiscaseE&AFsystemofthespectralproblem (1) – (2) and E&AFof the Samarskii-Ionkin problem (forming Riesz basis) differ from each other only in a finite number of first members. Consequently, the E&AF system of the spectral problem (1) - (2) also forms the Riesz basis in $L_2(0,1)$.

By*B*wedenoteasetoffunctions $q(x) \in L_2(0,1)$, representable in the form of a finite series (6), which is everywhere dense in $L_2(0,1)$. Thus, we formulate the main result of our paper:

Theorem 2.2. Let $q(x) \in L_2(0,1)$. Then E&AF system of the spectral problem (1) – (2) forms Riesz basis in $L_2(0,1)$and the set B is everywhere dense in $L_2(0,1)$.

Since the adjoint operators simultaneously possess the Riesz basis property of root functions, consequently, we get

Corollary 2.2. The set Bof functions $q(x) \in L_2(0,1)$, for which the E&AF system of the conjugate problem (1a) - (2b), that is, of the multiple differentiation operator L_1^* with integral perturbation of the first boundary value condition of the Samarskii-Ionkin problem, forms a Riesz basis in $L_2(0,1)$, is every where dense $L_2(0,1)$.

Previously, other approaches to the study of similar problems (1a) - (2b) with integral perturbation of the second boundary value condition were published in our papers [14], [15], [16].

The work paper [17], we prove uniqueness theorem, by one spectrum, for a Sturm-Liouville operator with non-separated boundary value conditions and a real continuous and symmetric potential.

Acknowledgments. These results are partially supported by Grant
№ AP05132587.

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ЕСЕЛІ ДИФФЕРЕНЦИАЛДАНАТЫН ЖҮКТЕЛГЕН ОПЕРАТОРДЫҢ ТҮБІРЛІК ВЕКТОРЛАР ЖҮЙЕСІНІҢ БАЗИСТІЛІГІ ЖАЙЛЫ

Аннотация. Бұл мақалада регулярлы, бірақ күшейтілген регулярлы емес шеттік шарттармен берілген жүктелген екінші ретті дифференциалдық оператордың спектралдық есебі қарастырылады. Еселі дифференциалданатын жүктелген оператордың түбірлік векторлар жүйесінің базистілігі зерттеледі. Кез келген өзіне-өзі түйіндес шеттік шарттармен және өзіне-өзі түйіндес формальді дифференциалдық амалмен берілген, спектрі дискретті болатын оператордың меншікті функциялары жүйесінің ортонормаланған базис құрайтындығы белгілі жай. Сонымен бірге өзіне-өзі түйіндес емес жай дифференциалдық операторлардың түбірлік функциялары жүйесінің базистілігіне шеттік шарттардан бөлек, дифференциалдық оператордың коэффициентерінің мәндері де әсер ететіндігі белгілі. Мұндай жағдайда, коэффициентердің мәндері шамалы ғана өзгергеннің өзінде, түбірлік функциялардың базистілік қасиеттеріне бірден әсер етеді. Тұңғыш рет мұндай факт В.А. Ильиннің жұмысында келтірілді. Өзіне-өзі түйіндес периодты есеп үшін В.А. Ильиннің идеясы өзіне-өзі түйіндес емес толқыту жағдайында А.С. Макиннің еңбегінде дамытылды. Бұл жұмыста, шеттік шарттардың біреуін толқытқан кезде оператор өзгерген болатын. Ал, М.А. Садыбеков пен Н.С. Иманбаевтың мақаласында периодты шарттармен берілген жүктелген екінші ретті дифференциалдық оператордың меншікті функциялары жүйесінің базистілік қасиеттері зерттелген. Бұл мақалада қарастырылған есепте өзіне-өзі түйіндес периодты есеп үшін өзіне-өзі түйіндес емес толқыту жағдайы болып табылады, бірақ А.С. Макиннің еңбегінде зерттелген есептен М.А. Садыбеков пен Н.С. Иманбаевтың мақаласында қарастырылған есептің ерекшелігі шеттік шарттардың емес, теңдеудің толқытылуында болып тұр.

Жүктелген дифференциалдық операторлардың түбірлік функцияларының базистілік қасиеттерін зерттеу мәселелері И.С. Ломовтың жұмыстарында зерттелді. В.А. Ильиннің спектралдық жіктеу әдісі И.С. Ломовтың мақалаларында жүктелген дифференциалдық операторлар үшін сәтті қолданылып дамытылды. А.М. Гомилко мен Г.В. Радзиевскийдің жұмыстарында функционалдық-дифференциалдық теңдеулердің түбірлік векторларының базистілік мәселелері басқа әдістермен зерттелген.

Ал, аталмыш мақалада екінші ретті, нөл нүктесінде жүктелінген дифференциалдық теңдеу үшін периодты емес шеттік шарттармен берілген есептің характеристикалық анықтауышы жазылып, оның спектралдық параметр бойынша бүтін аналитикалық функция болатындығы көрсетіліп, меншікті мәндері анықталған. Осыған сәйкес, түбірлік функцияларының базистілік қасиеттерінің орнықтылығы жайлы теорема дәлелденеді. Бұл есепке түйіндес есеп - толқытылған Самарский-Ионкин есебі болатындығы көрсетілген.

Түйін сөздер: меншікті мәндер, меншікті функциялар, тіркелген функциялар, түйіндес оператор, еселі дифференциалдау, жүктелген оператор, Рисс базистілігі, түбірлік векторлар.

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О БАЗИСНОСТИ СИСТЕМ КОРНЕВЫХ ВЕКТОРОВ НАГРУЖЕННОГО ОПЕРАТОРА КРАТНОГО ДИФФЕРЕНЦИРОВАНИЯ

Аннотация. В настоящей статье рассматривается возмущения дифференциального уравнения второго порядка спектральной задачи с нагруженным слагаемым, содержащий значение искомой функции в точке нуль, с регулярными, но неусиленно регулярными краевыми условиями. Исследуется вопрос базисности систем собственных и присоединенных функций (СиПФ) нагруженного оператора кратного дифференцирования. Хорошо известно, что система собственных функций оператора, заданного формально самосопряженным дифференциальным выражением, с произвольными самосопряженными краевыми условиями, обеспечивающими дискретный спектр, образует ортонормированный базис. Наряду с этим, известно, что в случае несамосопряженных обыкновенных дифференциальных операторов на базисность систем корневых функций, помимо краевых условий, могут влиять также значения коэффициентов дифференциального оператора. При этом базисные свойства корневых функций могут изменяться даже при сколь угодном малом изменении значений коэффициентов. Впервые этот факт был отмечен в работе В.А. Ильина. Идеи В.А. Ильина были развиты А.С. Макиным на случай несамосопряженного возмущения самосопряженной периодической

задачи. Оператор в работе А.С. Макина изменялся за счет возмущения одного из краевых условий. В статье М.А. Садыбекова, Н.С. Иманбаева исследованы базисные свойства корневых функций нагруженного дифференциального оператора второго порядка с периодическими краевыми условиями, который также является несамосопряженным возмущением самосопряженной периодической задачи. В отличие от работы А.С. Макина, в статье М.А. Садыбекова и Н.С. Иманбаева возмущение происходит за счет изменения уравнения. Вопросы базисности корневых функций нагруженных дифференциальных операторов были изучены в работах И.С. Ломова. Ему удалось распространить метод спектральных разложений В.А. Ильина на случай нагруженных дифференциальных операторов. Другим методом вопросы базисности функционально-дифференциальных уравнений были исследованы в работе А.М. Гомилко и Г.В. Радзиевского.

А в настоящей работе исследуется вопросы базисности корневых векторов дифференциального оператора второго порядка с нагруженным слагаемым в точке ноль с непериодическими краевыми условиями. Построен характеристический определитель, который является целой аналитической функцией. Доказана теорема об устойчивости свойства базисности корневых векторов и построен сопряженный оператор, который оказался возмущенной задачей Самарского-Ионкина.

Ключевые слова: собственные значение, собственные функций, присоединенные функции, сопряженный оператор, кратное дифференцирование, нагруженный оператор, базис Рисса, корневые вектора.

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