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RESEARCH OF MULTIPERIODIC SOLUTIONS OF PERTURBED LINEAR AUTONOMOUS SYSTEMS WITH DIFFERENTIATION OPERATOR ON THE VECTOR FIELD

Abstract. A linear system with a differentiation operator *D* with respect to the directions of vector fields of the form of the Lyapunov's system with respect to space independent variables and a multiperiodic toroidal form with respect to time variables is considered. All input data of the system multiperiodic depend on time variables or do not depend on them. The autonomous case of the system was considered in our early work. In this case, some input data received perturbations depending on time variables. We study the question of representing the required motion described by the system in the form of a superposition of individual periodic motions of rationally incommensurable frequencies. The initial problems and the problems of multiperiodicity of motions are studied. It is known that when determining solutions to problems, the system integrates along the characteristics outgoing from the initial points, and then, the initial data is replaced by the first integrals of the characteristic systems. Thus, the required solution consists of the following components: characteristics and first integrals of the characteristic systems of operator *D*, the matricant and the free term of the system itself. These components, in turn, have periodic and non-periodic structural components, which are essential in revealing the multiperiodic nature of the movements described by the system under study. The representation of a solution with the selected multiperiodic components is called the multiperiodic structure of the solution. It is realized on the basis of the well-known Bohr's theorem on the connection of a periodic function of many variables and a quasiperiodic function of one variable. Thus, more specifically, the multiperiodic structures of general and multiperiodic solutions of homogeneous and inhomogeneous systems with perturbed input data are investigated. In this spirit, the zeros of the operator D and the matricant of the system are studied. The conditions for the absence and existence of multiperiodic solutions of both homogeneous and inhomogeneous systems are established.
Keywords: multiperiodic solutions, autonomous system, operator of differentiation, Lyapunov's vector field, perturbation.

1. Introduction. The foundations of the method used in this note were laid in [1, 2], which were further developed in [3–14] and applied to the study of solutions different problems in the partial differential equations [15, 16]. These methods with simple modifications extend to the study solutions of problems of the differential and integro-differential equations of different types [1-16], in particular, problems on multi-frequency solutions of equations from control theory [17]. Many oscillatory phenomena are described by systems with a differentiation operator with respect to toroidal vector fields, and new methods based on the ideas of the Fourier [18], Poincaré-Lyapunov and Hamilton-Jacobi methods [19, 20] appear to establish their periodic oscillatory solutions. The methods of research for multiperiodic solutions are successfully combined by methods for studying solutions of boundary value problems for equations of mathematical physics. Elements of the methods of [1, 2] can easily be found in [21–25], where time-oscillating solutions of boundary value problems are studied by the parameterization method.

As noted above, the considered system of partial differential equations along with multidimensional time contains space independent variables, according to which differentiation is carried out to the directions of the different vector fields. The autonomous case of this system was considered in [15, 16], where differentiation with respect to time variables was carried out in the direction of the main diagonal of space, and the free term of the system was independent of time variables. In this case, these parameters of the systems received perturbations depending on time variables. In the note, the method for studying multiperiodic structures of general and multiperiodic solutions is developed, the conditions for the existence of a multiperiodic solution are established, and its integral representation is given.

We consider the system of linear equations

$$Dx = Ax + f(\tau, t, \zeta) \tag{1.1}$$

with differentiation operator

$$D = \frac{\partial}{\partial \tau} + \left\langle a, \frac{\partial}{\partial t} \right\rangle + \left\langle \nu I \zeta + g, \frac{\partial}{\partial \zeta} \right\rangle, \tag{1.2}$$

where $\tau \in (-\infty, +\infty) = R$, $t = (t_1, ..., t_m) \in R \times ... \times R = R^m$, $\zeta = (\zeta_1, ..., \zeta_l) \in R_\delta^{2l}$, $\zeta_j = (\xi_j, \eta_j)$, $j = \overline{1,l}$, $R_\delta^2 = \left\{ \zeta_j = (\xi_j, \eta_j) \in R^2 : |\zeta_j| = \sqrt{\xi_j^2 + \eta_j^2} < \delta, j = \overline{1,l} \right\}$,

$\delta = const > 0$ are independent variables with areas of change; $\frac{\partial}{\partial t} = \left(\frac{\partial}{\partial t_1}, ..., \frac{\partial}{\partial t_m} \right)$ and

$\frac{\partial}{\partial \zeta} = \left(\frac{\partial}{\partial \zeta_1}, ..., \frac{\partial}{\partial \zeta_l} \right)$, $\frac{\partial}{\partial \zeta_j} = \left(\frac{\partial}{\partial \xi_j}, \frac{\partial}{\partial \eta_j} \right)$, $j = \overline{1,l}$ are vector differentiation operators; $I = diag\,(I_2, ..., I_2)$ is a matrix with l-blocks, I_2 is symplectic unit of the second order, $\nu = (\nu_1, ..., \nu_l)$ is a constant vector, $\nu I = diag\,(\nu_1 I_2, ..., \nu_l I_2)$, $a = (a_1(\tau, t), ..., a_m(\tau, t)) = a(\tau, t)$, $g = (g_1(\tau), ..., g_l(\tau)) = g(\tau)$ are vector functions, $\langle \, , \, \rangle$ is the sign of the scalar product of vectors; A is a constant $n \times n$-matrix, $f = f(\tau, t, \zeta)$ is n-vector-function of variables $(\tau, t, \zeta) \in R \times R^m \times R_\delta^{2l}$.

The vector function $x(\tau, t, \zeta)$ is called (θ, ω)-periodic with respect to (τ, t) if the identity

$$x(\tau + \theta, t + q\omega, \zeta) = x(\tau, t, \zeta), \ (\tau, t, \zeta) \in R \times R^m \times R_\delta^{2l}, \ q \in Z^m,$$

was fulfilled, where $Z^m = Z \times ... \times Z$, Z is the set of integers, $\omega = (\omega_1, ..., \omega_m)$ is the vector-period, and the periods $\omega_0 = \theta, \omega_1, ..., \omega_m$ are rationally incommensurable positive constants:

$$q_j \omega_j + q_k \omega_k \neq 0, \ \ q_j, q_k \in Z, \ (j, k = \overline{0,m}).$$

The motion described by a (θ, ω)-periodic with respect to (τ, t) function $x = x(\tau, t, \zeta)$ is called a multiperiodic oscillation.

The main objective of this note is to determine the multiperiodic structures of solutions of the initial-multiperiodic problems associated with the system (1.1) - (1.2).

The objective was partially been touched upon by the authors in [15, 16], when the problem of multiperiod solutions of the autonomous system of the form (1.1) - (1.2) was considered, where time variables τ, t did not explicitly enter.

2. Multiperiodic structure of zeros of the differentiation operator D. We introduce the equation

$$Du = 0 \tag{2.1}$$

with the required scalar function $u = u(\tau, t, \zeta)$, where D is the differentiation operator with respect to (τ, t, ζ) of the form (1.1).

The solutions of equation (2.1) are called the zeros of the operator D .

Suppose that 1) the vector function $a(\tau,t)$ has the property of smoothness with respect to $(\tau,t)\in R\times R^m$ of order $(0,e)=(0,1,...,1)$:

$$a(\tau+\theta,t+q\omega)=a(\tau,t)\in C_{\tau,t}^{(0,e)}(R\times R^m),\ q\in Z^m, \tag{2.2}$$

2) positive constants $\nu_1, ..., \nu_l$ rationally incommensurable:

$$q_i\nu_i+q_j\nu_j\neq 0,\ q_i^2+q_j^2\neq 0,\quad q_i,q_j\in Z,\ (i,\ j=\overline{0,l}), \tag{2.3}$$

therefore, numbers $\alpha_j=2\pi\nu_j^{-1},\ j=\overline{1,l}$ are also incommensurable.

3) vector-functions $g_j(\tau)=(\varphi_j(\tau),\psi_j(\tau)),\ j=\overline{1,l}$ are continuous and β_j-periodic:

$$g_j(\tau+\beta_j)=g_j(\tau)\in C_\tau^{(0)}(R),\ j=\overline{1,l}, \tag{2.4}$$

where $\alpha_k,\ k=\overline{1,l}$ and $\beta_j,\ j=\overline{1,l}$ are incommensurable positive constants.

It follows from condition (2.2) that the vector field

$$\frac{dt}{d\tau}=a(\tau,t) \tag{2.5}$$

determines the characteristic

$$t=\lambda(\tau,\tau^0,t^0), \tag{2.5^1}$$

emanating from any initial point $(\tau^0,t^0)\in R\times R^m$, and moreover, it has the properties:

$$t^0=\lambda(\tau^0,\tau,t), \tag{2.5^2}$$

$$\lambda(\tau',\tau'',\lambda(\tau'',\tau,t))=\lambda(\tau',\tau,t),\ \tau',\tau''\in R, \tag{2.5^3}$$

$$\lambda(\tau^0+\theta,\tau+\theta,t+q\omega)=\lambda(\tau^0,\tau,t)+q\omega,\ q\in Z^m, \tag{2.5^4}$$

$$DV(\lambda(\tau^0,\tau,t))=0,\ V(t)\in C_t^{(e)}(R^m). \tag{2.5^5}$$

Obviously, $u=v(\lambda(\tau^0,\tau,t))$ satisfies the initial condition

$$u\big|_{\tau=\tau^0}=v(t)\in C_t^{(e)}(R^m). \tag{2.1'}$$

Properties (2.5²) - (2.5⁵) of the characteristic (2.5¹) of the vector field (2.5) are known from [2]. Hence, we will not dwell on their justification.

The solution

$$u(\tau^0,\tau,t)=v(\lambda(\tau^0,\tau,t)) \tag{2.6}$$

of the problem (2.1) - (2.1 ') is called the zero of the operator D with the initial condition (2.1').

Lemma 2.1. *Let condition (2.2) be satisfied. Then under the condition*

$$v(t+q\omega)=v(t)\in C_t^{(e)}(R^m),\ q\in Z^m \tag{2.7}$$

the zeros (2.6) of the operator D with the initial data (2.1') have the multiperiodicity property of the form

$$u(\tau^0+\theta,\tau+\theta,t+q\omega)=u(\tau^0,\tau,t),\ q\in Z^m. \tag{2.8}$$

The proof of identity (2.8) follows from the structure of zero (2.6), property (2.5⁴) which is a consequence of condition (2.2), and from condition (2.7).

Note that property (2.8) represent the diagonal θ-periodicity $u(\tau^0,\tau,t)$ with respect to (τ^0,τ) and ω-periodicity with respect to t.

In particular, when a function $\lambda(\tau^0,\tau,t)$ is θ-periodic with respect to τ or τ^0, then the zeros (2.6) of the operator D under the conditions of the lemma are (θ,ω) -periodic with respect to (τ,t).

The vector fields

$$\frac{d\zeta_j}{d\tau}=\nu_j I_2\zeta_j+g_j(\tau),\ j=\overline{1,l} \tag{2.9}$$

in scalar form have the form

$$\begin{cases}\dfrac{d\xi_j}{d\tau}=-\nu_j\eta_j+\varphi_j(\tau),\\ \dfrac{d\eta_j}{d\tau}=\nu_j\xi_j+\psi_j(\tau),\ \ j=\overline{1,l}.\end{cases} \tag{2.10}$$

Obviously, the matricants $Z_j(\tau),\ j=\overline{1,l}$ of the systems (2.10), and, consequently, the systems (2.9), are determined by periodic relations

$$Z_j(\tau)=\begin{pmatrix}\cos\nu_j\tau & -\sin\nu_j\tau\\ \sin\nu_j\tau & \cos\nu_j\tau\end{pmatrix},\ j=\overline{1,l} \tag{2.11}$$

with periods $\alpha_j=2\pi\nu_j^{-1},\ j=\overline{1,l}$. The conditions

$$\det\left[Z_j(\beta_j)-Z_j(0)\right]\neq 0,\ j=\overline{1,l}. \tag{2.12}$$

are satisfied by virtue the incommensurability α_k and β_j. Indeed

$$\det\left[Z_j(\beta_j)-Z_j(0)\right]=2\left(1-\cos\nu_j\beta_j\right)\neq 0\ ,$$

since $\beta_j-q_j\alpha_j\neq 0,\ \ j=\overline{1,l}$.

Then systems (2.9) allow for β_j -periodic solutions

$$z_j(\tau)=\left[Z_j^{-1}(\tau+\beta_j)-Z_j^{-1}(\tau)\right]^{-1}\int_{\tau}^{\tau+\beta_j}Z_j^{-1}(s)\,g_j(s)ds,\ j=\overline{1,l}. \tag{2.13}$$

Consequently, the general solutions ζ_j of the systems (2.9) have the form

$$\zeta_j=Z_j\left(\tau-\tau^0\right)\left[\zeta_j^0-z_j\left(\tau^0\right)\right]+z_j\left(\tau\right),\ j=\overline{1,l}, \tag{2.14}$$

where the matricants $Z_j\left(\tau\right),\ j=\overline{1,l}$ and solutions $z_j(\tau),\ \ j=\overline{1,l}$ have periodicity properties

$$Z_j\left(\tau+\alpha_j\right)=Z_j\left(\tau\right),\ \ j=\overline{1,l}, \tag{2.15}$$

$$z_j\left(\tau+\beta_j\right)=z_j\left(\tau\right),\ \ j=\overline{1,l}. \tag{2.16}$$

We must introduce new time variables $s_j,\ \sigma_j, j=\overline{1,l}$ and space variables $h_j, j=\overline{1,l}$ related by relations

$$h_j\left(s_j-s_j^0,\sigma_j,\ \zeta_j^0-z_j^0\right)=Z_j\left(s_j-s_j^0\right)\left[\zeta_j^0-z_j^0\right]+z_j\left(\sigma_j\right),\ \ j=\overline{1,l}, \tag{2.17}$$

in order to represent solutions (2.14) using periodic functions with incommensurable periods $\alpha_j,\ \beta_j, j=\overline{1,l}$, where $z_j^0=z_j\left(s_j^0\right)$, s_j^0 are the initial values of the variables $s_j,\ j=\overline{1,l}$.

Obviously, the multiperiodic functions (2.17) present the solutions (2.14) under $\sigma_j=s_j=\tau,\ \ s_j^0=\tau^0$, moreover, they satisfy equations

$$\frac{\partial h_j}{\partial s_j}+\frac{\partial h_j}{\partial \sigma_j}=\nu_j I_2 h_j+g_j(\sigma_j)\quad j=\overline{1,l} \tag{2.18}$$

with the initial conditions

$$h_j\big|_{\sigma_j=s_j=s_j^0}=\zeta_j^0,\ \ j=\overline{1,l}. \tag{2.18°}$$

By virtue of the properties (2.15) and (2.16) of the matricants $Z_j(\tau)$ and the solutions $z_j(\tau)$, the functions (2.17) have the properties of multi-periodicity

$$h_j\left(s_j+\alpha_j,\sigma_j,\zeta_j^0\right)=h_j\left(s_j,\sigma_j+\beta_j,\zeta_j^0\right)=h_j\left(s_j,\sigma_j,\zeta_j^0\right),\ j=\overline{1,l}. \tag{2.19}$$

Thus, we obtained from systems of equations (2.9) to systems of equations (2.18) with initial conditions (2.18°) by introducing new time variables.

We get the equations (2.9) and their solutions (2.14) from the systems of equations (2.18) - (2.18°) by substitution $\sigma_j=s_j=\tau,\ \ s_j^0=\tau^0$ conversely.

The close relationship between the functions $\sigma_j=\sigma_j(\tau)$ and $h_j=h_j(s_j,\sigma_j)$ of the form

$$\sigma_j(\tau)=h_j(\tau,\tau),\quad \frac{d\sigma_j}{d\tau}=\frac{dh_j(\tau,\tau)}{d\tau}=\frac{\partial h_j(s_j,\sigma_j)}{\partial s_j}+\frac{\partial h_j(s_j,\sigma_j)}{\partial \sigma_j}$$

with $\sigma_j=s_j=\tau$ leads to a transition from the differentiation operator D to the differentiation operator

$$\overline{D}=\frac{\partial}{\partial\tau}+\left\langle a(\tau,t),\frac{\partial}{\partial t}\right\rangle+\left\langle e,\frac{\partial}{\partial s}\right\rangle+\left\langle e,\frac{\partial}{\partial \sigma}\right\rangle+\left\langle \nu I h+g(\sigma),\frac{\partial}{\partial h}\right\rangle+\left\langle \frac{\partial h}{\partial s}+\frac{\partial h}{\partial \sigma},\frac{\partial}{\partial h}\right\rangle, \tag{2.20}$$

where $s=(s_1,...,s_l),\sigma=(\sigma_1,...,\sigma_l), e=(1,...,1)$ – l-vector, $h=(h_1,...,h_l),\ \ h_j=h_j(s_j,\sigma_j),$ $j=\overline{1,l},\ \frac{\partial h}{\partial s}=\left(\frac{\partial h_1}{\partial s_1},...,\frac{\partial h_l}{\partial s_l}\right),\ \frac{\partial h}{\partial \sigma}=\left(\frac{\partial h_1}{\partial \sigma_1},...,\frac{\partial h_l}{\partial \sigma_l}\right).$

Further, we obtain the characteristic

$$\zeta=Z(\tau-\tau^0)\left[\zeta^0-z(\tau^0)\right]+z(\tau) \tag{2.21}$$

of the matrix-vector equation

$$\frac{d\zeta}{d\tau}=\nu I\zeta+g(\tau), \tag{2.22}$$

which is characteristic for equation (2.1) with respect to space variables, based on the coordinate data (2.9) - (2.16), where $Z(\tau)=diag\ \left[Z_1(\tau),...,Z_l(\tau)\right], z(\tau)=\left(z_1(\tau),...,z_l(\tau)\right),\ \zeta^0=\left(\zeta_1^0,...,\zeta_l^0\right).$

We have the first integral

$$\zeta^0=Z(\tau^0-\tau)\left[\zeta-z(\tau)\right]+z(\tau^0)\equiv\mu(\tau^0,\tau,\zeta) \tag{2.23}$$

of equation (2.22) from the equation of characteristic (2.21).

Therefore, we obtain the identity

$$D\mu(\tau^0,\tau,\zeta)=0,\quad \mu(\tau^0,\tau^0,\zeta)=\zeta. \tag{2.24}$$

Then we have the solution

$$u(\tau^0,\tau,\zeta)=w(\mu(\tau^0,\tau,\zeta)), \tag{2.25}$$

of equation (2.1) satisfying the initial condition

$$u\big|_{\tau=\tau^0}=w(\zeta)\in C_\zeta^{(e)}(R^l). \tag{2.1"}$$

for any differentiable function $w(\zeta)\in C_\zeta^{(e)}(R^l)$.

Indeed, since $Du = \dfrac{\partial w}{\partial \zeta} \cdot D\mu$, by virtue of (2.24) we have $Du = 0$. Thus, (2.25) with the condition (2.1") is the zero of the operator D.

Further, we have a vector function

$$h\left(s - s^0, z(\sigma), \zeta^0 - z^0\right) = Z\left(s - s^0\right)\left[\zeta^0 - z\left(s^0\right)\right] + z(\sigma), \tag{2.26}$$

satisfying the characteristic equation of the operator $\overline{D}$ of the form

$$\frac{\partial h}{\partial s} + \frac{\partial h}{\partial \sigma} = \nu I h + g(\sigma) \tag{2.27}$$

with the initial condition

$$h\Big|_{\sigma = s = s^0} = \zeta^0,$$

based on our analysis related to relations (2.17) - (2.19) for studying the multi-periodic structure of characteristic (2.23), where $g(\sigma) = \left(g_1(\sigma_1), ..., g_l(\sigma_l)\right)$, $z(\sigma) = \left(z_1(\sigma_1), ..., z_l(\sigma_l)\right)$, $Z(s) = diag\ \left[Z_1(s_1), ..., Z_l(s_l)\right]$, $h = (h_1, ..., h_l)$, $h_j = h_j\left(s_j - s_j^0, z(\sigma_j^0), \zeta_j^0 - z(s^0)\right)$, $j = \overline{1, l}$, $\dfrac{\partial h}{\partial s} = \left(\dfrac{\partial h_1}{\partial s_1}, ..., \dfrac{\partial h_l}{\partial s_l}\right)$, $\dfrac{\partial h}{\partial \sigma} = \left(\dfrac{\partial h_1}{\partial \sigma_1}, ..., \dfrac{\partial h_l}{\partial \sigma_l}\right)$.

Obviously, by virtue properties (2.15), (2.16) and (2.19), the matrix $Z(s)$ is periodic with period $\alpha = (\alpha_1, ..., \alpha_l)$, and the solution $z(\sigma)$ with period $\beta = (\beta_1, ..., \beta_l)$.

The first integral of the equation (2.27) is determined from the equation of characteristic (2.26) by the relation

$$\zeta^0 = h\left(s^0 - s, z(s^0), \zeta - z(\sigma)\right).$$

It's obvious that

$$\overline{D} h\left(s^0 - s, z(s^0), \zeta - z(\sigma)\right) = 0, \ \ h\Big|_{\sigma = s = s^0} = \zeta. \tag{2.28}$$

Moreover, we have

$$\overline{D} w\left(h\left(s^0 - s, z(s^0), \zeta - z(\sigma)\right)\right) = \frac{\partial w(h)}{\partial \zeta} \cdot \overline{D} h\left(s^0 - s, z(s^0), \zeta - z(\sigma)\right) = 0,$$

for any differentiable function $w(\zeta)$, by virtue of (2.28), at that

$$w\left(h\left(s^0 - s, z(s^0), \zeta - z(\sigma)\right)\right)\Big|_{\sigma = s = s^0} = w(\zeta).$$

Thus,

$$\overline{u}\left(s^0, s, \sigma, \zeta\right) = w\left(h\left(s^0 - s, z(s^0), \zeta - z(\sigma)\right)\right) \tag{2.29}$$

is the zero of the operator $\overline{D}$, that under $\sigma = s = \tau\tilde{e}$, $s^0 = \tau^0\tilde{e}$ it becomes the $u\left(\tau^0, \tau, \zeta\right)$ zero of the operator D, where $\tilde{e} = (1, ..., 1)$ is a l-vector.

Lemma 2.2. *Let conditions (2.3) and (2.4) be satisfied. Then the zeros (2.25) of the operator* D *with the initial condition (2.1") have a multiperiodic structure of the form (2.29) with the vector function (2.26), at that*

$$\begin{gathered} \overline{u}\left(s^0, s, \sigma, \zeta\right)\Bigg|_{\substack{\sigma = s = \tau\tilde{e} \\ s^0 = \tau^0\tilde{e}}} = u\left(\tau^0, \tau, \zeta\right), \\ h\left(\tilde{e}\tau^0 - \tilde{e}\tau, z(\tilde{e}\tau^0), \zeta - z(\tilde{e}\tau)\right) = \mu\left(\tau^0, \tau, \zeta\right). \end{gathered} \tag{2.30}$$

Theorem 2.1. *Let conditions (2.2) - (2.4) be satisfied. Then the solution* $u\left(\tau^0,\tau,t,\zeta\right)$ *of equation (2.1) with the initial condition*

$$u\Big|_{\tau=\tau^0} = u^0(t,\zeta)\in C_{t,\zeta}^{(\hat{e},\tilde{e})}\left(R^m\times R^l\right) \tag{2.1°}$$

is determined by the relation

$$u\left(\tau^0,\tau,t,\zeta\right)= u^0\left(\lambda\left(\tau^0,\tau,t\right),\mu\left(\tau^0,\tau,\zeta\right)\right), \tag{2.31}$$

which under the conditions

$$\lambda\left(\tau^0,\tau+\theta,t\right)=\lambda\left(\tau^0,\tau,t\right), \tag{2.32}$$

$$u^0(t+q\omega,\zeta)=u^0(t,\zeta),\ \ q\in Z^m \tag{2.33}$$

has a multiperiodic structure with respect to (τ,t,s,σ) *with period* $(\theta,\omega,\alpha,\beta)$ *of the form*

$$\overline{u}\left(\tau^0,\tau,t;s^0,s,\sigma,\zeta\right)= u^0\left(\lambda\left(\tau^0,\tau,t\right),h\left(s^0-s,z\left(s^0\right),\zeta-z(\sigma)\right)\right), \tag{2.34}$$

where the vector-function $h(s,z,\zeta)$ *has the form (2.26),* $\hat{e}=(1,...,1)$ *is* m*-vector,* $\tilde{e}=(1,...,1)$ *is* l*-vector, moreover*

$$\overline{u}\Big|_{\substack{\sigma=s=\tilde{e}\tau\\ s^0=\tilde{e}\tau^0}} = u\left(\tau^0,\tau,t,\zeta\right). \tag{2.35}$$

Proof. The form of solution (2.31) of the initial problem (2.1) - (2.1°) follows from the general theory of the first-order partial differential equations. Special cases of it are given in Lemmas 2.1 and 2.2.

The multiperiodic structure (2.34) of the solution (2.31) is also contained in the indicated lemmas; and the multiperiodicity is easily verified under the additional conditions (2.32) and (2.33).

The statement (2.35) follows from (2.30).

Note that, $\overline{u}=\overline{u}\left(\tau^0,\tau,t,s^0,s,\sigma,\zeta\right)$ is the solution of the equation $\overline{D}\overline{u}=0$ with the differentiation operator $\overline{D}$.

The proved theorem is the multiperiodic structure of the zeros of the differentiation operator D.

In conclusion, we note that if the conditions (2.32) and (2.33) do not fulfill, then the representation (2.34) remains the multi-periodic structure of the solution (2.31). But then a definite structure (2.34) does not possess the periodicity property with respect to τ,t.

3. The multiperiodic structure of the solution of a homogeneous linear D **-system with constant coefficients.** We consider a homogeneous linear system

$$Dx = Ax \tag{3.1}$$

with a differentiation operator D of the form (1.2) and a constant $n\times n$-matrix A.

We will put the problem of determining the multiperiodic structure of the solution x of the system (3.1) with the initial condition

$$x\Big|_{\tau=\tau^0} = u(t,\zeta)\in C_{t,\zeta}^{(\hat{e},\tilde{e})}\left(R^m\times R^l\right). \tag{3.1°}$$

To this end, we begin the solution of the problem by studying the multiperiodic structure of the matricant

$$X(\tau)=\exp\left[A\,\tau\right] \tag{3.2}$$

of the system (3.1).

We need the following lemmas to that end.

Lemma 3.1. *If* $f_j\left(\tau+\theta_j\right)=f_j(\tau),\ j=\overline{1,r}$ *is some collection of the periodic functions with rationally commensurate periods:* $\theta_j/\theta_k=r_{jk}$ *is a rational number for* $j,k=\overline{1,r}$*, then for these functions exist a common period* θ*:*

$$f_j(\tau+\theta)=f_j(\tau),\ j=\overline{1,r}.$$

Indeed, by virtue of rational commensurability exist integer natural numbers $q_1,..., q_r$ such that $q_1\theta_1 = ... = q_r\theta_r = \theta$, which is the required period.

Lemma 3.2. *If the real parts of all eigenvalues equal to zero and all the elementary divisors are simple of the constant matricant* $Y(\tau) = \exp[I\tau]$, *then all the elements of the matrix* I *are periodic functions*.

Proof. By the conditions of the Lemma 3.2, the eigenvalues are $\lambda_j(I) = ib_j$, $j = \overline{1, r}$, where $i = \sqrt{-1}$ is the imaginary unit; the constants b_j are either equal to zero or nonzero. If it is nonzero, then each eigenvalue $\lambda_j(I) = ib_j$ corresponds to one or more Jordan cells J_j of the form

$$J_j = \begin{pmatrix} 0 & -b_j \\ b_j & 0 \end{pmatrix}.$$

Then the matricant has the form

$$Y(\tau) = K\, diag\left[e^{I_1\tau},..., e^{I_r\tau}\right]K^{-1}, \tag{3.3}$$

where if $b_j = 0$, then $I_j = 0$ and if $b_j \neq 0$, then $I_j = J_j$, moreover

$$Y_j(\tau) = e^{J_j\tau} = \begin{pmatrix} \cos b_j\tau & -\sin b_j\tau \\ \sin b_j\tau & \cos b_j\tau \end{pmatrix}, \; (b_j \neq 0), \tag{3.4}$$

K is a matrix of reduction I to the actual canonical form $I = K\, diag\,[I_1,..., I_r]K^{-1}$.

We have a complete proof of the Lemma 3.2 from relations (3.3) and (3.4), and the periods of the elements of the matrix $Y(\tau)$ are determined as $\gamma_1 = 2\pi b_{j_1}^{-1},..., \gamma_\rho = 2\pi b_{j_\rho}^{-1}$ on the basis of the Lemma 3.1, taking into account the commensurability of the periods $2\pi b_j^{-1}$, $j = \overline{1, r}$, $\rho \leq r$ Periods $\gamma_1,...,\gamma_\rho$ are rationally incommensurable constants.

Further, cells $Y_{j_k}(\tau)$, $j_k = \overline{1, r_k}$ of the form (3.4) having the periodicity property with a period γ_k will be considered as cells depending on the variable $\tau = \tau_k$:

$$Y_{j_k}(\tau_k + \gamma_k) = Y_{j_k}(\tau_k), \; j_k = \overline{1, r_k}. \tag{3.5}$$

Representing each cell (3.4) using the new variables $\tau_1,..., \tau_\rho$ in accordance with condition (3.5), from the expression of the matricant (3.3) we obtain a multiperiodic matrix $T(\hat{\tau}) = T(\tau_1,..., \tau_\rho)$ with period $\gamma = (\gamma_1,..., \gamma_\rho)$.

Since

$$\frac{\partial}{\partial \tau_k} Y_{j_k}(\tau_k) = J_j Y_{j_k}(\tau_k),$$

the matrix $T(\hat{\tau})$ satisfies the equation

$$\hat{D}T(\hat{\tau}) = I\, T(\hat{\tau}), \tag{3.6}$$

where the operator $\hat{D}$ is determined by

$$\hat{D} = \left\langle \hat{e}, \frac{\partial}{\partial \hat{\tau}} \right\rangle = \frac{\partial}{\partial \tau_1} + ... + \frac{\partial}{\partial \tau_\rho}, \tag{3.7}$$

$\hat{e} = (1,...,1)$ is a ρ-vector.

Obviously, under $\hat{\tau} = \hat{e}\,\tau$ we have $T(\hat{e}\,\tau) = Y(\tau)$ and

$$\frac{d}{d\tau}Y(\tau)=\frac{d}{d\tau}T(\hat{e}\,\tau)=I\,T(\hat{e}\,\tau)=I\,Y(\tau). \tag{3.8}$$

Thus, the multiperiodic matrix $T(\hat{\tau})$ defines the multiperiodic structure of the matricant $Y(\tau)$:

$$Y(\tau)=T(\tau_1,...,\tau_\rho)\Big|_{\tau_1=...=\tau_\rho=\tau}. \tag{3.9}$$

Lemma 3.3. *The matricant $Y(\tau)$ of the system (3.8) under the conditions of Lemma 3.2 has a multiperiodic structure in the form of a matrix $T(\hat{\tau})=T(\tau_1,...,\tau_\rho)$, which satisfies the system (3.6) with the differentiation operator (3.7) and along the characteristics $\hat{\tau}=\hat{e}\,\tau$ of the operator $\hat{D}$ turns into $Y(\tau)$, in other words, these matrices are related by the relation (3.9).*

It's known that from the course of linear algebra the matrix A can be represented in the form

$$A=K\,J(\lambda)K^{-1}=K\,J(a+ib)K^{-1}=K\,J(a)K^{-1}+K\,E(ib)K^{-1}=R+I,$$

where K is some non-singular matrix for reducing the matrix A to Jordan normal form $J(\lambda)=diag\,[J_1(\lambda_1),...,J_r(\lambda_r)]$ with Jordan's n_j-cells $J_j(\lambda_j)$ corresponding to eigenvalues $\lambda_j=a_j+ib_j$, $j=\overline{1,r}$; $R=K\,J(a)K^{-1}$ is the matrix, $J(a)$ is matrix obtained from the Jordan form $J(\lambda)$ by replacing the eigenvalues λ_j with their real parts $a_j=\mathrm{Re}\,\lambda_j$, $j=\overline{1,r}$, $I=K\,E(ib)K^{-1}$ is the matrix, $E(ib)=diag[ib_1E_1,...,ib_rE_r]$, $b_j=\mathrm{Im}\lambda_j$, $j=\overline{1,r}$, E_j is the unit n_j-cells, $j=\overline{1,r}$, moreover, the matrices R and I are commutative: $RI=IR.$. Therefore, $e^{A\tau}=e^{I\tau+R\tau}=e^{I\tau}\cdot e^{R\tau}$, otherwise, the matricant (3.2) can be represented as

$$X(\tau)=Y(\tau)\cdot Z(\tau), \tag{3.10}$$

where $Y(\tau)=\exp[I\tau], Z(\tau)=\exp[R\,\tau]$, moreover, along with property (3.8), $Y(\tau)$ satisfies the equation

$$\frac{d}{d\tau}Y(\tau)=AY(\tau)-Y(\tau)R. \tag{3.11}$$

Indeed, we making the replacement

$$X=Y(\tau)Z$$

in the equation

$$\dot{X}=AX \tag{3.12}$$

obtain the equation

$$\dot{Z}=Y^{-1}(\tau)\left[AY(\tau)-\frac{d}{d\tau}Y(\tau)\right]Z.$$

Then, we obtain the identity (3.10) taking into account that $\dot{Z}=RZ$, where $Z(\tau)=\exp[R\,\tau]$.

The identities (3.8) and (3.11) establish the connection of the matricant $Y(\tau)=\exp[I\tau]$ with the triple of matrices A,R,I; moreover, the matrix I satisfies the conditions of Lemma 3.2. Therefore, according to Lemma 3.3, the multiperiodic structure of the matricant $X(\tau)=\exp[A\,\tau]$, by virtue of equality (3.10), is determined by a matrix $\hat{X}(\tau,\hat{\tau})$ of the form

$$\hat{X}(\tau,\hat{\tau})=X(\tau,\tau_1,...,\tau_\rho)=T(\tau_1,...,\tau_\rho)e^{R\tau}, \tag{3.13}$$

which is connected by the matricant $X(\tau)$, by relation

$$\hat{X}(\tau,\hat{\tau})\Big|_{\hat{\tau}=\hat{e}\tau} = X(\tau). \tag{3.14}$$

Thus, the following theorem is proved.

Theorem 3.1. *In the presence of complex eigenvalues of the matrix A, the matricant (3.2) of the system (3.12) has a multiperiodic structure defined by the matrix (3.13) and relations (3.6) - (3.9), and it along the characteristics $\hat{\tau} = \hat{e}\tau$ of the operator $\hat{D}$ satisfies condition (3.14). The matrix $T(\hat{\tau})$ turns into a constant matrix in the absence of complex eigenvalues.*

Now the solution of the objectives set can be formulated as Theorem 3.2.

Theorem 3.2. *Let conditions (2.2) - (2.4) be satisfied. Then the solution $x(\tau^0,\tau,t,\zeta)$ of the problem (3.1) - (3.1°) defined by relation*

$$x(\tau^0,\tau,t,\zeta) = X(\tau)u(\lambda(\tau^0,\tau,t),\mu((\tau^0,\tau,\zeta))) \tag{3.15}$$

has a multi-periodic structure in the form of a vector-function

$$\hat{x}(\tau^0,\tau,\hat{\tau},t,\ s^0 s,\sigma,\zeta) = \hat{X}(\tau,\hat{\tau})\overline{u}(\lambda(\tau^0,\tau,t),h(s^0-s,z(s^0),\zeta-z(\sigma))), \tag{3.16}$$

that satisfies equation

$$\overline{\overline{D}}\hat{x} = A\hat{x} \tag{3.17}$$

with the differentiation operator

$$\overline{\overline{D}} = \overline{D} + \hat{D}, \tag{3.18}$$

defined by relations (2.20) and (3.7).

Proof. The representation (3.15) is known from [2], and (3.16) follows from the proved Theorems 2.1 and 3.1. The identity (3.17) can be verified by a simple check.

Now we investigate the question of the existence of nonzero multiperiodic solutions of the systems of equations (3.1).We begin the study with the simplest cases.

We consider a canonical system with a single zero eigenvalue

$$\frac{dx_1}{d\tau}=0,\ \frac{dx_2}{d\tau}=x_1,\ldots,\frac{dx_n}{d\tau}=x_{n-1},$$

which in the vector-matrix form has the form

$$\frac{dx}{d\tau}=E_* x, \tag{3.19}$$

where E_* is the sub-diagonal unit oblique series of the n-th order, $x=(x_1,\ldots,x_n)$.

We introduce a triangular matrix $X_0(\tau)$ with elements of the form of power functions:

$$X_0(\tau)=\begin{pmatrix} 1 & 0 & \ldots & 0 \\ \tau & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ \dfrac{\tau^{n-1}}{(n-1)!} & \dfrac{\tau^{n-2}}{(n-2)!} & \ldots & 1 \end{pmatrix}$$

and an arbitrary constant vector $c=(c_1,\ldots,c_n)$ to represent the general solution x of the system (3.19).

Then the general solution of the system (3.19) is represented in the form $x = X_0(\tau)c$.

It easy to see from the structure of the general solution that system (3.19) admits a one-parameter family of periodic solutions x^* of the form

$$x^*(\tau)=X_0(\tau)c^*, \tag{3.20}$$

where $c^*=(0,\ldots,0,c_n^*)$, c_n^* is an arbitrary parameter.

Next, we consider a system of pairs $\left(x_j', x_j''\right)$ of equations of the form

$$\frac{dx_1'}{d\tau} = -bx_1'', \ \frac{dx_1''}{d\tau} = bx_1', \ \frac{dx_j'}{d\tau} = x_{j-1}' - bx_j'', \ \frac{dx_j''}{d\tau} = x_{j-1}'' + bx_j', \ j = \overline{1,l},$$

which can be represented using the vector $x_j = \left(x_j', x_j''\right)$ in the form

$$\frac{dx_1}{d\tau} = bI_2 x_1, \ \frac{dx_j}{d\tau} = E_2 x_j + bI_2 x_j, \ j = \overline{1,l},$$

where E_2 is the second-order identity matrix, I_2 is the second-order symplectic identity matrix, $b = const \neq 0$.

If we introduce a constant block matrix

$$J(b) = \begin{pmatrix} bI_2 & O & O & \dots & O & O & O \\ E_2 & bI_2 & O & \dots & O & O & O \\ O & E_2 & bI_2 & \dots & O & O & O \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ O & O & O & \dots & E_2 & bI_2 & O \\ O & O & O & \dots & O & E_2 & bI_2 \end{pmatrix}$$

with blocks I_2, E_2 and second-order zero blocks O, then the system under consideration with a vector $x = (x_1, \dots, x_l)$ can be represented in the form

$$\frac{dx}{d\tau} = J(b)\,x, \tag{3.21}$$

which we call a canonical system with a single pair of purely imaginary conjugate eigenvalues $\lambda = (ib, -ib)$.

We introduce a diagonal block matrix

$$T^*(\tau) = diag\,\left[T_2(\tau), \dots, T_2(\tau)\right]$$

with a block $T_2^*(\tau)$ of the form

$$T_2^*(\tau) = \begin{pmatrix} \cos b\tau & -\sin b\tau \\ \sin b\tau & \cos b\tau \end{pmatrix}$$

and a triangular block matrix with elements of the form of power functions:

$$Y^*(\tau) = \begin{pmatrix} E_2 & O & \dots & O \\ \tau E_2 & E_2 & \dots & O \\ \dots & \dots & \dots & \dots \\ \dfrac{\tau^{l-1}}{(l-1)!}E_2 & \dfrac{\tau^{l-2}}{(l-2)!}E_2 & \dots & E_2 \end{pmatrix}$$

to represent the general solution x of the system (3.21).

Then the matricant $X^*(\tau)$ of the system (3.21) can be represented as $X^*(\tau) = T^*(\tau)\,Y^*(\tau)$, and the general solution $x(\tau)$ is determined by the relation

$$x(\tau) = X^*(\tau)\,c$$

with an arbitrary constant vector $c = (c_1, \ldots, c_l),\ c_j = (c'_j,\, c''_j),\ j = \overline{1,l}$.

We obtain easily a family of $\theta = 2\pi b^{-1}$-periodic solutions $x^*(\tau)$ by parameters c'_l and c''_l of the form

$$x^*(\tau) = X^*(\tau)\,c^* \tag{3.22}$$

with a constant vector $c^* = (0, \ldots, 0, c_l), c_l = (c'_l,\, c''_l)$ from the structure of the general solution

Now, by replacing $x = Kz$ with a non-singular constant matrix K, we reduce the system (3.1) to the canonical form

$$Dz = J(A)z,\ J(A) = K^{-1}AK, \tag{3.1'}$$

which consists of subsystems in accordance with Jordan's cells of the matrix A.

Obviously, systems (3.1) and (3.1') are equivalent with respect to the existence of multiperiodic solutions.

It is also clear that the system (3.1') has subsystems of the form

$$Dz_1 = E_* z_1, \tag{3.1'_1}$$

or

$$Dz_2 = J(b)z_2, \tag{3.1'_2}$$

respectively with matrices similar to the matrices of systems (3.19) and (3.21), in the presence of zero or purely imaginary eigenvalue. Obviously, nonzero solutions of (3.20) and (3.22) satisfy the systems $(3.1'_1)$ and $(3.1'_2)$, respectively

Consequently, in the cases under consideration, system (3.1') allows nonzero periodic solutions $z^*(\tau)$. Then $Kz^*(\tau) = x^*(\tau)$ is a periodic solution of the system (3.1).

Thus, the following theorem is proved.

Theorem 3.3. *Under the conditions of the Theorem 3.2, the system (3.1) allowed nonzero multiperiodic solutions enough for the matrix* A *to have at least one eigenvalue* $\lambda = \lambda(A)$ *with the real part* $\operatorname{Re}\lambda(A) = 0$ *equal to zero.*

We have the following theorem from the theorem 3.3, as a corollary.

Theorem 3.4. *Under the conditions of the Theorem 3.3, the system (3.1) did not admit the multiperiodic solution other than trivial, it is sufficient that all eigenvalues of the matrix* A *have nonzero real parts.*

Since the system (3.1) is (θ, ω)-periodic, of particular interest is the question of the existence of its nonzero multiperiodic solutions with the same periods.

The general solution x of the system (3.1) can be represented in the form

$$x(\tau, t, \zeta) = X(\tau)\,u(\tau, t, \zeta), \tag{3.23}$$

where $u = u(\tau, t, \zeta)$ is the zero of the operator D with the general initial condition for $\tau = 0$

$$x(0, t, \zeta) = u(0, t, \zeta) = u_0(t, \zeta),$$

$X(\tau) = \exp[A\,\tau]$ is the matricant of the system.

Among the zeros of the operator D there exist multiperiodic ones, in particular, constants by the Theorem 2.1.

Theorem 3.5. *Under the conditions (2.2) - (2.4), the system (3.1) had (θ, ω) -periodic with respect to (τ, t) solutions of the form (3.23) corresponding to the multiperiodic zero of the operator D with the same periods, it is necessary and sufficient that the monodromy matrix $X(\theta)$ satisfies condition*

$$\det\left[X(\theta) - E\right] = 0. \tag{3.24}$$

Proof. Under the conditions of the theorem, its justice is equivalent to the solvability of equation

$$X(\tau + \theta)u = X(\tau)u \tag{3.25}$$

in the space of (θ, ω) -periodic with respect to (τ, t) zeros $u = u(\tau, t, \zeta)$ of the operator D.

We arrive at the solvability of the system of equations

$$\left[X(\theta) - E\right]u = 0,$$

which is equivalent to the condition (3.24) taking into account the properties of the matricant $X(\tau + \theta) = X(\tau)X(\theta)$ from the system (3.25).

In conclusion, we note that the fulfillment of condition

$$\det\left[X(\theta) - E\right] \neq 0 \tag{3.26}$$

guarantees the absence of such solutions.

We also note that condition (3.24) is a sufficient sign of the existence of the nonzero multiperiodic solution of the system (3.1).

Theorem 3.6. Let conditions (2.2) - (2.4) and (3.26) be satisfied. Then the system (3.1) allowed nonzero (θ, ω) -periodic solutions of the form (3.23) necessary and sufficient for the functional-difference equations

$$u(\tau + \theta, t + q\omega, \zeta) = \left[X(\theta) - E\right]^{-1} X(\theta)\left[u(\tau + \theta, t + q\omega, \zeta) - u(\tau, t, \zeta)\right], \; q \in Z^m \tag{3.27}$$

to be solvable in the space of zeros of the operator D.

Proof. Under the condition (3.26) from the definition of (θ, ω) -periodicity with respect to (τ, t) of solution (2.23), we have the equation (3.27). We must be to take into account that $u(\tau, t, \zeta)$ is the zero of the operator D to complete the proof.

If the equation (3.27) has only zero solutions, then, under the condition (3.26), the system (3.1) does not have a nontrivial multiperiodic solution.

We also note that the fulfillment of the condition

$$\operatorname{Re}\lambda_j(A) \neq 0, \; j = \overline{1, n}$$

on the non-zero real parts $\operatorname{Re}\lambda_j(A)$ of all eigenvalues $\lambda_j(A)$ of the matrix A ensures the fulfillment of condition (3.26).

In conclusion, we note that on the basis of the multiperiodic structures (2.30) and (3.13) the characteristics $\mu(\tau^0, \tau, \zeta)$ of the matricant $X(\tau)$ and by the theorems which proved above, it is easy to obtain structures of (θ, ω) -periodic with respect to (τ, t) solutions of the system (3.1) expressed in terms of variables $\tau, \hat{\tau}, s, \sigma, t, \zeta$.

4. The multiperiodic structure of an inhomogeneous linear system with operator D. Consider the inhomogeneous linear equation

$$Dx = Ax + f(\tau, t, \zeta) \tag{4.1}$$

corresponding to the homogeneous equation (3.1), where the n-vector function $f(\tau, t, \zeta)$ satisfies condition

$$f(\tau + \theta, t + q\omega, \zeta) = f(\tau, t, \zeta) \in C^{(0, \hat{e}, \tilde{e})}_{\tau, t, \zeta}\left(R \times R^m \times R^l\right). \tag{4.2}$$

Assume that the condition (3.26) is fulfilled and we search for the (θ,ω)-periodic with respect to (τ,t) solution $x(\tau,t,\zeta)$ of the system (4.1) that corresponds to zero $u(\tau,t,\zeta)$ of the operator D possessing the property of multiperiodicity with the same periods (θ,ω) for (τ,t).

Therefore, we have the solution

$$x(\tau,t,\zeta)=X(\tau)u(\tau,t,\zeta)+X(\tau)\int_0^{\tau}X^{-1}(s)f(s,\lambda(s,\tau,t),\mu(s,\tau,\zeta))ds \quad (4.3)$$

with zero $u(\tau+\theta,t+q\omega,\zeta)=u(\tau,t,\zeta),\ q\in Z^m$ of the operator D having the property $x(\tau+\theta,t+q\omega,\zeta)=x(\tau,t,\zeta),\ q\in Z^m$.

Then the solution (4.3) has another representation

$$x(\tau,t,\zeta)=X(\tau+\theta)u(\tau,t,\zeta)+X(\tau+\theta)\int_0^{\tau+\theta}X^{-1}(s)f(s,\lambda(s,\tau+\theta,t),\mu(s,\tau+\theta,\zeta))ds. \quad (4.4)$$

Further, we obtain

$$x(\tau,t,\zeta)=\left[X^{-1}(\tau+\theta)-X^{-1}(\tau)\right]^{-1}\left[\int_0^{\tau+\theta}X^{-1}(s)f(s,\lambda(s,\tau+\theta,t),\mu(s,\tau+\theta,\zeta))ds+\right.$$

$$\left.+\int_{\tau}^{0}X^{-1}(s)f(s,\lambda(s,\tau,t),\mu(s,\tau,\zeta))ds\right],$$

(4.5)

eliminating from identities (4.3) and (4.4) the unknown zero $u(\tau,t,\zeta)$ of the operator D, where the reversible of the matrix $\left[X^{-1}(\tau+\theta)-X^{-1}(\tau)\right]$ follows from condition (3.26).

If we accept the notation

$$f_\theta(s,\lambda(s,\tau,t),\mu(s,\tau,\zeta))=\begin{cases}f(s,\lambda(s,\tau,t),\mu(s,\tau,\zeta)),\ \tau\xrightarrow{s}0,\\ f(s,\lambda(s,\tau+\theta,t),\mu(s,\tau+\theta,\zeta)),\ 0\xrightarrow{s}\tau+\theta,\end{cases}$$

then formula (4.5) can be represented in a more compact form

$$x(\tau,t,\zeta)=\left[X^{-1}(\tau+\theta)-X^{-1}(\tau)\right]^{-1}\int_{\tau}^{\tau+\theta}X^{-1}(s)f_\theta(s,\lambda(s,\tau,t),\mu(s,\tau,\zeta))ds. \quad (4.6)$$

where $\gamma\xrightarrow{s}\delta$ means changes in the variable s from γ to δ. Obviously, if the system (3.1) does not have multiperiodic solutions, except for zero, then the solution (4.6) of the system (4.1) is a unique multiperiodic solution.

Further, we have solutions

$$\hat{x}(s,\sigma,\hat{\tau},\tau,t,\zeta)=\left[\hat{X}^{-1}(\tau+\theta,\hat{\tau}+\hat{e}\theta)-\hat{X}^{-1}(\tau,\hat{\tau})\right]^{-1}\times$$
$$\times\int_{\tau}^{\tau+\theta}X^{-1}(\varepsilon)f_\theta(\varepsilon,\lambda(\varepsilon,\tau,t),h(\varepsilon-s,z(\varepsilon),\zeta-z(\sigma)))d\varepsilon \quad 4.7)$$

of the equation

$$\overline{\overline{D}}\hat{x}=A\hat{x}+f(\tau,t,\zeta)$$

with the differentiation operator (3.18) from representation (4.6) on the basis of multiperiodic structures (2.30) and (3.13) of the quantity $\mu(s,\tau,\zeta)$ and $X(\tau)$.

Thus, the following theorem is proved.

Theorem 4.1. *Assume that conditions (2.2) - (2.4), (3.26) and (4.2) are satisfied, and the homogeneous system (3.1) does not have multiperiodic solutions except zero. Then the system (4.1) has a unique* (θ,ω)*-periodic solution (4.6) for which the* $(\alpha,\beta,\gamma,\theta,\omega)$*-periodic with respect to* $(s,\sigma,\hat{\tau},\tau,t)$ *structure (4.7) satisfies equation (4.8) with the differentiation operator (3.18).*

In conclusion, note that we can derive the multiperiodic structure of the general solution (4.3) of the system (4.1) similarly to formula (4.7).

Conclusion. A method for studying the multiperiodic structure of oscillatory solutions of perturbed linear autonomous systems of the form (1.1) - (1.2) was developed. The main essence of the method for studying the multiperiodic structures of solution of the system under consideration is a combination of the known methods [1-3] with the methods used in [15, 16] for the autonomous systems. In this case, some system input received perturbations depending on the time variables *τ, t*. In conclusion, the sufficient conditions for the existence of the multiperiodic solutions of linear systems (1.1) - (1.2) with the differentiation operator *D* in the directions of a toroidal vector field with respect to time variables and of the form of Lyapunov's systems with respect to space variables were established. Moreover, relation (4.6) is an integral representation of the multiperiodic solution of the system, and (4.7) determines its multiperiodic structure. We also note that the integral representation given here differs from the analogue given in [15, 16].

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ВЕКТОРЛЫҚ ӨРІС БОЙЫНША ДИФФЕРЕНЦИАЛДАУ ОПЕРАТОРЛЫ ҚОЗДЫРЫЛҒАН СЫЗЫҚТЫ АВТОНОМДЫҚ ЖҮЙЕЛЕРДІҢ КӨППЕРИОДТЫ ШЕШІМДЕРІН ЗЕРТТЕУ

Аннотация. Тәуелсіз кеңістік айнымалысына қатысты Ляпунов жүйесі түріндегі және уақыт айнымалысына қатысты көппериодты тороидалды түрдегі векторлық өрістер бағыты бойынша *D* дифференциалдау операторлы сызықты жүйе қарастырылады. Жүйені анықтайтын барлық берілгендер не уақыт айнымалысынан көппериодты тәуелді, не олардан тәуелсіз болады. Жүйенің автономдық жағдайы бұрынғы жұмыстарда қарастырылған. Бұл жағдайда жүйені анықтайтын кейбір берілгендерге уақыт айнымалысынан тәуелді қоздыртқы берілген. Рационалды өлшенбейтін жиіліктердің жекеленген периодты қозғалыстарының суперпозициясы түріндегі жүйе арқылы сипатталған ізделінді қозғалыс туралы сұрақ зерттеледі. Бастапқы есептер және қозғалыстардың көппериодтылығы туралы есептер зерттеледі. Есептің шешімін анықтау кезінде жүйе бастапқы нүктеден шығатын характеристика маңайында интегралданатыны, одан кейін бастапқы берілгендер характеристикалық жүйенің бірінші интегралдарымен ауыстырылатыны белгілі. Сонымен ізделінді шешім келесі компоненттерден тұрады: *D* операторының характеристикалық жүйесінің характеристикасы мен бірінші интегралдары, жүйенің бос мүшесі мен матриცанты. Бұл компоненттердің зерттелуші жүйемен сипатталған қозғалыстың көппериодтылық табиғатын ашу кезінде маңызды мағынасы бар болатын периодты және периодты емес құрылымдық құраушылары болады. Шешімді ерекшеленген көппериодты құраушылар арқылы сипаттауды шешімнің көппериодтылық құрылымы деп аталған. Ол көп айнымалылы периоды функциялар мен бір айнымалылы квазипериодты функцияларының байланысы туралы Бордың танымал теоремасы негізінде жүзеге асады. Сонымен, жүйелерді анықтайтын берілгендері қоздырылған жағдайда біртекті және біртексіз жүйелердің жалпы және көппериодты шешімдерінің көппериодты құрылымы нақты зерттелген. Осылайша *D* операторының нөлдері мен жүйенің матриცанты зерттелген. Біртекті және біртексіз жүйелердің көппериодты шешімдерінің бар болуы және болмауы шарттары тағайындалған.

Түйін сөздер: Көппериодты шешім, автономдық жүйе, дифференциалдау операторы, Ляпунов векторлық өрісі, қоздыртқы.

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ИССЛЕДОВАНИЕ МНОГОПЕРИОДИЧЕСКИХ РЕШЕНИЙ ВОЗМУЩЕННЫХ ЛИНЕЙНЫХ АВТОНОМНЫХ СИСТЕМ С ОПЕРАТОРОМ ДИФФЕРЕНЦИРОВАНИЯ ПО ВЕКТОРНОМУ ПОЛЮ

Аннотация. Рассматривается линейная система с оператором дифференцирования D по направлениям векторных полей вида системы Ляпунова относительно пространственных независимых переменных и многопериодического тороидального вида относительно временных переменных. Все входные данные системы либо многопериодично зависят от временных переменных, либо от них не зависят. Автономный случай системы рассмотрен в нашей ранней работе. В данном случае некоторые входные данные получили возмущения, зависящие от временных переменных. Исследуется вопрос о представлении искомого движения, описанного системой в виде суперпозиции отдельных периодических движений рационально несоизмеримых частот. Изучаются начальные задачи и задачи о многопериодичности движений. Известно, что при определении решений задач система интегрируется вдоль характеристик, исходящих из начальных точек, а затем, начальные данные заменяются первыми интегралами характеристических систем. Таким образом, искомое решение состоит из следующих компонентов: характеристик и первых интегралов характеристических систем оператора D, матриканта и свободного члена самой системы. Эти компоненты, в свою очередь, имеют периодические и непериодические структурные составляющие, которые имеют существенное значение при раскрытии многопериодической природы движений, описанных исследуемой системой. Представление решения с выделенными многопериодическими составляющими названо многопериодической структурой решения. Оно реализуется на основе известной теоремы Бора о связи периодической функции от многих переменных и квазипериодической функции одной переменной. Таким образом, более конкретно, исследуются многопериодические структуры общих и многопериодических решений однородных и неоднородных систем с возмущенными входными данными. В таком духе изучаются нули оператора D и матрикант системы. Устанавливаются условия отсутствия и существования многопериодических решений как однородных, так и неоднородных систем.

Ключевые слова: Многопериодическое решение, автономная система, оператор дифференцирования, Ляпунова векторное поле, возмущение.

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