

ISSN 2518-1726 (Online),  
ISSN 1991-346X (Print)

ҚАЗАҚСТАН РЕСПУБЛИКАСЫ  
ҰЛТТЫҚ ҒЫЛЫМ АКАДЕМИЯСЫНЫҢ  
Әль-фараби атындағы Қазақ ұлттық университетінің

# Х А Б А Р Л А Р Ы

---

---

## ИЗВЕСТИЯ

НАЦИОНАЛЬНОЙ АКАДЕМИИ НАУК  
РЕСПУБЛИКИ КАЗАХСТАН  
Қазақстан Республикасының  
Ғылым Академиясының  
Әль-Фараби атындағы  
Қазақ ұлттық университетінің

## NEWS

OF THE NATIONAL ACADEMY OF SCIENCES  
OF THE REPUBLIC OF KAZAKHSTAN  
Al-farabi kazakh  
national university

**SERIES**  
**PHYSICO-MATHEMATICAL**

**5 (327)**

**SEPTEMBER-OCTOBER 2019**

PUBLISHED SINCE JANUARY 1963

PUBLISHED 6 TIMES A YEAR

ALMATY, NAS RK

Б а с р е д а к т о р ы  
ф.-м.ғ.д., проф., ҚР ҰҒА академигі **Ғ.М. Мұтанов**

Р е д а к ц и я а л қ а с ы:

**Жұмаділдаев А.С.** проф., академик (Қазақстан)  
**Кальменов Т.Ш.** проф., академик (Қазақстан)  
**Жантаев Ж.Ш.** проф., корр.-мүшесі (Қазақстан)  
**Өмірбаев У.У.** проф. корр.-мүшесі (Қазақстан)  
**Жүсіпов М.А.** проф. (Қазақстан)  
**Жұмабаев Д.С.** проф. (Қазақстан)  
**Асанова А.Т.** проф. (Қазақстан)  
**Бошкаев К.А.** PhD докторы (Қазақстан)  
**Сұраған Д.** корр.-мүшесі (Қазақстан)  
**Quevedo Hernando** проф. (Мексика),  
**Джунушалиев В.Д.** проф. (Қырғыстан)  
**Вишневский И.Н.** проф., академик (Украина)  
**Ковалев А.М.** проф., академик (Украина)  
**Михалевич А.А.** проф., академик (Белорус)  
**Пашаев А.** проф., академик (Әзірбайжан)  
**Такибаев Н.Ж.** проф., академик (Қазақстан), бас ред. орынбасары  
**Тигиняну И.** проф., академик (Молдова)

**«ҚР ҰҒА Хабарлары. Физика-математикалық сериясы».**

**ISSN 2518-1726 (Online), ISSN 1991-346X (Print)**

Меншіктенуші: «Қазақстан Республикасының Ұлттық ғылым академиясы» РҚБ (Алматы қ.)  
Қазақстан республикасының Мәдениет пен ақпарат министрлігінің Ақпарат және мұрағат комитетінде  
01.06.2006 ж. берілген №5543-Ж мерзімдік басылым тіркеуіне қойылу туралы куәлік

Мерзімділігі: жылына 6 рет.  
Тиражы: 300 дана.

Редакцияның мекенжайы: 050010, Алматы қ., Шевченко көш., 28, 219 бөл., 220, тел.: 272-13-19, 272-13-18,  
<http://physics-mathematics.kz/index.php/en/archive>

---

© Қазақстан Республикасының Ұлттық ғылым академиясы, 2019

Типографияның мекенжайы: «Аруна» ЖК, Алматы қ., Муратбаева көш., 75.

Главный редактор  
д.ф.-м.н., проф. академик НАН РК **Г.М. Мутанов**

Редакционная коллегия:

**Джумадильдаев А.С.** проф., академик (Казахстан)  
**Кальменов Т.Ш.** проф., академик (Казахстан)  
**Жангаев Ж.Ш.** проф., чл.-корр. (Казахстан)  
**Умирбаев У.У.** проф., чл.-корр. (Казахстан)  
**Жусупов М.А.** проф. (Казахстан)  
**Джумабаев Д.С.** проф. (Казахстан)  
**Асанова А.Т.** проф. (Казахстан)  
**Бошкаев К.А.** доктор PhD (Казахстан)  
**Сураган Д.** чл.-корр. (Казахстан)  
**Quevedo Hernando** проф. (Мексика),  
**Джунушалиев В.Д.** проф. (Кыргызстан)  
**Вишневский И.Н.** проф., академик (Украина)  
**Ковалев А.М.** проф., академик (Украина)  
**Михалевич А.А.** проф., академик (Беларусь)  
**Пашаев А.** проф., академик (Азербайджан)  
**Такибаев Н.Ж.** проф., академик (Казахстан), зам. гл. ред.  
**Тигиняну И.** проф., академик (Молдова)

«Известия НАН РК. Серия физико-математическая».

ISSN 2518-1726 (Online), ISSN 1991-346X (Print)

Собственник: РОО «Национальная академия наук Республики Казахстан» (г. Алматы)

Свидетельство о постановке на учет периодического печатного издания в Комитете информации и архивов  
Министерства культуры и информации Республики Казахстан №5543-Ж, выданное 01.06.2006 г.

Периодичность: 6 раз в год.

Тираж: 300 экземпляров.

Адрес редакции: 050010, г. Алматы, ул. Шевченко, 28, ком. 219, 220, тел.: 272-13-19, 272-13-18,  
<http://physics-mathematics.kz/index.php/en/archive>

---

© Национальная академия наук Республики Казахстан, 2019

Адрес типографии: ИП «Аруна», г. Алматы, ул. Муратбаева, 75.

E d i t o r i n c h i e f  
doctor of physics and mathematics, professor, academician of NAS RK **G.M. Mutanov**

E d i t o r i a l b o a r d:

**Dzhumadildayev A.S.** prof., academician (Kazakhstan)  
**Kalmenov T.Sh.** prof., academician (Kazakhstan)  
**Zhantayev Zh.Sh.** prof., corr. member. (Kazakhstan)  
**Umirbayev U.U.** prof. corr. member. (Kazakhstan)  
**Zhusupov M.A.** prof. (Kazakhstan)  
**Dzhumabayev D.S.** prof. (Kazakhstan)  
**Asanova A.T.** prof. (Kazakhstan)  
**Boshkayev K.A.** PhD (Kazakhstan)  
**Suragan D.** corr. member. (Kazakhstan)  
**Quevedo Hernando** prof. (Mexico),  
**Dzhunushaliyev V.D.** prof. (Kyrgyzstan)  
**Vishnevskiy I.N.** prof., academician (Ukraine)  
**Kovalev A.M.** prof., academician (Ukraine)  
**Mikhalevich A.A.** prof., academician (Belarus)  
**Pashayev A.** prof., academician (Azerbaijan)  
**Takibayev N.Zh.** prof., academician (Kazakhstan), deputy editor in chief.  
**Tiginyanu I.** prof., academician (Moldova)

**News of the National Academy of Sciences of the Republic of Kazakhstan. Physical-mathematical series.**

**ISSN 2518-1726 (Online), ISSN 1991-346X (Print)**

Owner: RPA "National Academy of Sciences of the Republic of Kazakhstan" (Almaty)

The certificate of registration of a periodic printed publication in the Committee of information and archives of the Ministry of culture and information of the Republic of Kazakhstan N 5543-Ж, issued 01.06.2006

Periodicity: 6 times a year

Circulation: 300 copies

Editorial address: 28, Shevchenko str., of. 219, 220, Almaty, 050010, tel. 272-13-19, 272-13-18,  
<http://physics-mathematics.kz/index.php/en/archive>

---

© National Academy of Sciences of the Republic of Kazakhstan, 2019

Address of printing house: ST "Aruna", 75, Muratbayev str, Almaty

## NEWS

OF THE NATIONAL ACADEMY OF SCIENCES OF THE REPUBLIC OF KAZAKHSTAN  
PHYSICO-MATHEMATICAL SERIES

ISSN 1991-346X

<https://doi.org/10.32014/2019.2518-1726.56>

Volume 5, Number 327 (2019), 19 – 39

UDK 517.929

A.Sh. Shaldanbayev<sup>1</sup>, S.M. Shalenova<sup>2</sup>, M.B. Ivanova<sup>3</sup>, A.A. Shaldanbayeva<sup>4</sup>

<sup>1</sup>Silkway International University, Shymkent, Kazakhstan;<sup>2</sup>Yessenov University; Aktau, Kazakhstan;<sup>3</sup>South Kazakhstan medical Academy, Shymkent, Kazakhstan;<sup>4</sup>Regional social-innovative University, Shymkent, Kazakhstan[shaldanbaev51@mail.ru](mailto:shaldanbaev51@mail.ru), [salima\\_shalenova@mail.ru](mailto:salima_shalenova@mail.ru), [marina-iv@mail.ru](mailto:marina-iv@mail.ru), [altima\\_a@mail.ru](mailto:altima_a@mail.ru)

## ON SPECTRAL PROPERTIES OF A BOUNDARY VALUE PROBLEM OF THE FIRST ORDER EQUATION WITH DEVIATING ARGUMENT

**Abstract.** In this paper, we study spectral properties of a boundary value problem of a first order differential equation with constant coefficients and deviating argument; the deviation is present at the highest term of the equation, and it cannot be transferred to the lower terms of the equation without an additional condition. By spectral properties, we mean completeness and basic properties of a system of eigenfunctions and associated functions of a boundary value problem, as well as Volterra properties.

**Keywords:** equation with deviating argument, completeness, basic property, Volterra property, Gaal's formula, Lidsky's theorem, Sturm – Liouville operator, Riesz basis.

**1. Introduction.** The first works on the theory of differential equations with involution are found in the scientific literature of the nineteenth century. This is the extensive work of Babbage [1] from 1816. The work consists of several parts. Algebraic and transcendental equations with involution, integral and differential equations containing an involution are considered. At present, mathematicians from many countries are studying differential equations with involution. The reference devoted to the study of differential equations with involution is quite extensive. An extensive bibliography contains monographs by D. Przevorska-Rolewicz [2], J. Wiener [3], Jack K. Hale Sjoerd M. Verduyn Lunel [4].

Partial differential equations with involution arise in mathematical models of population dynamics, ecology and physiology. A series of papers by S. Busenberg [6], J.M. Cushing [7-8] and others, devoted to mathematical modeling in population theory (biology), suggest the need for deep research on the analytical theory of differential equations with involution. In the work of J. Wiener [3, p. 264], attempts were made to apply the method of variable separation to partial differential equations with involution. In this case, solution is sought as a series in eigenfunctions. Conditions on existence of unbounded solutions of the considered problems are obtained, as well as the condition for a series to diverge in terms of eigenfunctions. Among the studies of recent years we can note the work of W. Watkins [9-10], which deals with solvability of one-dimensional differential equations with involution, and A.P. Khromov and his followers [11–12], which consider questions of solvability of integral and partial differential equations with involution.

Method of variable separation for solving partial differential equations is based on the spectral theory of one-dimensional differential operators. Spectral theory of self-adjoint and non-self-adjoint ordinary differential operators, which originated in depths of the equations of mathematical physics and began with classical works of Sturm, Liouville, Steklov and others, has received a fairly complete development over the past century. Spectral theory of self-adjoint ordinary differential operators is almost complete. In the field of spectral theory of non-self-adjoint ordinary differential operators, substantial results on completeness and basicity of eigenfunctions and associated functions are obtained by M.V. Keldysh [13],

V.A. Il'in [14-19], M. Otelbaev [20], A.A. Shkalikov [21], Radzievsky [22] and many other mathematicians.

Theory of basicity of systems of eigenfunctions and associated functions of non-self-adjoint ordinary differential operators, proposed by V.A. Ilyin, received rapid development. Review papers [23-24] give a fairly complete picture of development of the basicity theory by V.A. Il'in.

Compared with the spectral theory of ordinary differential operators, the spectral theory of one-dimensional differential operators with involution is in its infancy. Apparently, the first works on the spectral theory of one-dimensional differential operators with involution were carried out on initiative of T.Sh. Kalmenov [25-29] in the past decade of this century. These studies were continued in the cycle of works by M.A. Sadybekov and A.M. Sarsenby [30-35]. Over the past decade, interest of researchers to differential equations with involutions has noticeably increased, as evidenced by the publications [36-52]. The bases theory is described in [54-55] in detail.

In this paper we continue the studies begun in [28], a brief summary of this paper was announced in [53].

**Formulation of the problem.** We investigate the boundary value problem

$$Ly = ay'(x) + by'(1-x) = \lambda y(x), x \in (0,1), \quad (1.1)$$

$$\alpha y(0) + \beta y(1) = 0, \quad (1.2)$$

where  $a, b, \alpha, \beta$  are arbitrary, previously known constants, and  $\lambda$  is a spectral parameter,  $y(x)$  is a desired function from the class  $C^1(0,1) \cap C[0,1]$ .

Let operator  $P$  be defined by the following formula

$$Py(x) = ay(x) + by(1-x),$$

then

$$P^{-1}v(x) = \frac{(aI - bS)v(x)}{a^2 - b^2},$$

where  $Sv(x) = v(1-x)$ ,  $I$  is unit operator.

Indeed,

$$\begin{aligned} PP^{-1}v(x) &= (aI + bS) \frac{(aI - bS)v(x)}{a^2 - b^2} = \\ &= \frac{a^2I - abS + baS - b^2S^2}{a^2 - b^2} = \frac{(a^2 - b^2)I}{a^2 - b^2} = I. \end{aligned}$$

Consequently, if  $a^2 - b^2 \neq 0$ , then the problem (1.1) - (1.2) takes the following form:

$$\begin{aligned} Ly &= P \frac{d}{dx} y(x) = f(x), x \in (0,1), \\ \alpha y(0) + \beta y(1) &= 0. \end{aligned}$$

$$\begin{cases} y'(x) = \lambda P^{-1}y(x), \\ \alpha y(0) + \beta y(1) = 0; \end{cases} \begin{cases} y'(x) = \lambda \frac{aI - bS}{a^2 - b^2} y(x), \\ \alpha y(0) + \beta y(1) = 0. \end{cases}$$

It is a generalized spectral problem.

Let  $Qy(x) = ay(x) + by(1-x)$ , then

$$Q^{-1}v(x) = \frac{(aI - bS)v(x)}{a^2 - \beta^2}.$$

Supposing  $Qy(x) = z(x)$ , we have

$$y(x) = Q^{-1}z(x) = \frac{aI - bS}{a^2 - \beta^2} z(x) = \frac{\alpha z(x) - \beta z(1-x)}{a^2 - \beta^2},$$

$$y'(x) = \frac{\alpha z'(x) + \beta z'(1-x)}{\alpha^2 - \beta^2} = \frac{Qz'(x)}{\alpha^2 - \beta^2};$$

Then our problem takes the following form:

$$\begin{aligned} \frac{Qz'(x)}{\alpha^2 - \beta^2} &= \lambda \frac{aI - bS}{\alpha^2 - b^2} Q^{-1}z(x), \Rightarrow Qz'(x) = \lambda \frac{\alpha^2 - \beta^2}{\alpha^2 - b^2} Q^{-1}z(x), \\ Qz'(x) &= \lambda(\alpha^2 - \beta^2)P^{-1}Q^{-1}z(x), \\ \begin{cases} z'(x) = \lambda(\alpha^2 - \beta^2)Q^{-1}P^{-1}Q^{-1}z(x), \\ z(0) = 0. \end{cases} & \end{aligned} \quad (1.3), (1.4)$$

It is a generalized spectral Cauchy problem.

**Remark 1.1.** If  $(\alpha^2 - \beta^2)(\alpha^2 - b^2) \neq 0$ , then the problem (1.1)-(1.2) is equivalent to the generalized spectral Cauchy problem (1.3)-(1.4).

## 2. Research Methods.

**2.1. Solvability.** If  $y_n(x) = \sin n\pi x$ , then

$$\begin{aligned} y_n(1-x) &= (-1)^{n+1}y_n(x), y_{2m}(1-x) = -y_{2m}(x), \\ y_{2m-1}(x) &= y_{2m-1}(1-x); \end{aligned}$$

moreover  $y_n(0) = 0, y_n(1) = 0$ .

a)  $y_{2m}(1-x) + y_{2m}(x) = 0, \Rightarrow y'_{2m}(x) - y'_{2m}(1-x) = 0;$

б)  $y_{2m-1}(x) - y_{2m-1}(1-x) = 0, \Rightarrow y'_{2m-1}(x) + y'_{2m-1}(1-x) = 0.$

**Lemma 2.1.** If  $Ly_0 = 0, y_0(x) \neq 0$ , then

$$(\alpha^2 - b^2)(\alpha + \beta) = 0. \quad (2.1)$$

**Proof.** From the equation (1.1) we have

$$\begin{cases} \alpha y'(x) + \beta y'(1-x) = 0, \\ \beta y'(x) + \alpha y'(1-x) = 0; \end{cases} \Rightarrow$$

a) or  $\alpha^2 - b^2 = 0$ , and  $y'(x) \neq 0$ ;

б) or  $\alpha^2 - b^2 \neq 0$ , and  $y'(x) \equiv 0$ .

If  $a = b \neq 0$ , then the problem has the form

$$\begin{cases} y'(x) + y'(1-x) = 0, \\ \alpha y(0) + \beta y(1) = 0. \end{cases}$$

Functions  $y_{2m-1}(x) = \sin(2m-1)\pi x, m = 1, 2, \dots$  are solutions of this problem, therefore, in this case the point  $\lambda_0 = 0$  is an infinitely multiple eigenvalue of the boundary value problem (1.1)-(1.2).

If  $b = -a \neq 0$ , then the boundary value problem (1.1)-(1.2) takes the following form

$$\begin{cases} y'(x) - y'(1-x) = 0, \\ \alpha y(0) + \beta y(1) = 0. \end{cases}$$

Solution of this problem is the following functions

$$y_{2m}(x) = \sin 2m\pi x, m = 1, 2, \dots$$

therefore, also in this case the point  $\lambda_0 = 0$  is infinitely multiple eigenvalue of the boundary value problem (1.1) - (1.2).

$$\begin{aligned} \text{If } \alpha^2 - b^2 \neq 0, \text{ then } y'(x) &\equiv 0, y(x) = C - \text{const}; \Rightarrow \\ \alpha \cdot C + \beta \cdot C = 0, (\alpha + \beta) \cdot C &= 0, \Rightarrow \alpha + \beta = 0, \end{aligned}$$

since  $C \neq 0$ .

Let  $(\alpha^2 - b^2)(\alpha + \beta) = 0$ , then

a) or  $\alpha + \beta = 0$ , or  $\alpha + \beta \neq 0$ , then  $\alpha^2 - b^2 = 0$ .

If  $\alpha + \beta = 0$ , then the function  $y_0(x) = C - const$  is an eigenfunction and  $\lambda_0 = 0$  is an eigenvalue.

If  $a^2 - b^2 = 0$ , then as we have already noted, in the case  $a = b$  the functions  $y_{2m-1}(x) = \sin(2m-1)\pi x, m = 1, 2, \dots$  are eigenfunctions, and  $\lambda_0 = 0$  is eigenvalue of the infinitely multiple. In the case  $b = -a$  the functions  $y_{2m}(x) = \sin 2m\pi x$  are eigenfunctions, and  $\lambda_0 = 0$  is infinitely multiple eigenvalue.

**Lemma 2.2.** Operator  $L$  is invertible if and only if

$$(a^2 - b^2)(\alpha + \beta) \neq 0, \quad (2.2)$$

where

$$Ly = ay'(x) + by'(1-x), \quad (1.1)$$

$$\alpha y(0) + \beta y(1) = 0. \quad (1.2)$$

**Proof.** If  $(a^2 - b^2)(\alpha + \beta) = 0$ , then in the case  $a^2 - b^2 = 0, \lambda_0 = 0$  is eigenvalue, thus the operator  $L$  is not invertible, and in the case  $a^2 - b^2 \neq 0, \lambda_0 = 0$  is a simple eigenvalue, therefore the operator  $L$  is again non-invertible.

Inversely, if operator  $L$  is non-invertible, then there exists a function  $D_0(x) \neq 0$  such that  $Ly_0 = 0$ , then due to Lemma 2.1, we have the equality (2.1).

## 2.2. About inverse operator.

$$Ly = ay'(x) + by'(1-x) = f(x), x \in (0,1), \quad (1.1)$$

$$\alpha y(0) + \beta y(1) = 0, |\alpha| + |\beta| \neq 0. \quad (1.2)$$

Let  $Tu(x) = au(x) + bu(1-x) = (aI + bS)u(x)$ ,

where  $Su(x) = u(1-x)$ , then

$$T^{-1}v(x) = \frac{aI - bS}{a^2 - b^2}v(x).$$

Indeed,

$$TT^{-1}v(x) = (aI + bS) \frac{(aI - bS)v(x)}{a^2 - b^2} = \frac{(a^2I - abS + baS - b^2S^2)}{a^2 - b^2}v(x) = v(x).$$

The problem (1.1)-(1.2) takes the following form

$$\begin{cases} T \frac{d}{dx} y(x) = f(x), x \in (0,1), \\ \alpha y(0) + \beta y(1) = 0. \end{cases}$$

Consequently,

$$\begin{aligned} y'(x) &= T^{-1}f(x), \Rightarrow \\ y(x) &= y(0) + \int_0^x T^{-1}f(t)dt, y(1) - y(x) = \int_x^1 T^{-1}f(t)dt, \\ y(x) &= y(1) - \int_x^1 T^{-1}f(t)dt, \Rightarrow \\ (\alpha + \beta)y(x) &= \alpha \int_0^x T^{-1}f(t)dt - \beta \int_x^1 T^{-1}f(t)dt, \end{aligned}$$



$$\begin{aligned}
y(x) &= L^{-1}f(x) = \frac{\alpha}{\alpha + \beta} \int_0^x T^{-1}f(t)dt - \frac{\beta}{\alpha + \beta} \int_x^1 T^{-1}f(t)dt = \\
&= \frac{\alpha}{\alpha + \beta} \int_0^1 \theta(x-t)T^{-1}f(t)dt - \frac{\beta}{\alpha + \beta} \int_0^1 \eta(x-t)T^{-1}f(t)dt = \\
&= \int_0^1 \frac{\alpha\theta(x-t) - \beta\theta(t-x)}{\alpha + \beta} T^{-1}f(t)dt = \\
&= \int_0^1 \frac{\alpha\theta(t-x) - \beta\theta(t-x)}{\alpha + \beta} \cdot \frac{af(t) - bf(1-t)}{a^2 - b^2} dt = \\
&= \frac{a}{(a^2 - b^2)(\alpha + \beta)} \int_0^1 [\alpha\theta(x-t) - \beta\theta(t-x)]f(t)dt - \\
&\quad - \frac{b}{(a^2 - b^2)(\alpha + \beta)} \int_0^1 [\alpha\theta(x-t) - \beta\theta(t-x)]f(t)dt = \\
&\quad \frac{a}{(a^2 - b^2)(\alpha + \beta)} \int_0^1 [\alpha\theta(x-t) - \beta\theta(t-x)]f(t)dt - \\
&\quad - \frac{b}{(a^2 - b^2)(\alpha + \beta)} \int_0^1 [\alpha\theta(x-1+t) - \beta\theta(1-t-x)]f(t)dt = \\
&= \int_0^1 \frac{a[\alpha\theta(x-t) - \beta\theta(t-x)] - b[\alpha\theta(x-1+t) - \beta\theta(1-t-x)]}{(a^2 - b^2)(\alpha + \beta)} \cdot \\
&\quad \cdot f(t)dt = \int_0^1 K(x,t)f(t)dt,
\end{aligned}$$

where

$$K(x,t) = \frac{a[\alpha\theta(x-t) - \beta\theta(t-x)] - b[\alpha\theta(x-1+t) - \beta\theta(1-t-x)]}{(a^2 - b^2)(\alpha + \beta)}.$$

We have proved the following theorem.

**Lemma 2.3.** If

$$(a^2 - b^2)(\alpha + \beta) \neq 0$$

then the inverse operator  $L^{-1}$  exists and has the form

$$y(x) = L^{-1}f(x) = \int_0^1 K(x,t)f(t)dt,$$

where

$$K(x,t) = \frac{a[\alpha\theta(x-t) - \beta\theta(t-x)] - b[\alpha\theta(x-1+t) - \beta\theta(1-t-x)]}{(a^2 - b^2)(\alpha + \beta)}.$$

### 2.3. Criteria about Volterra property.

We consider the following boundary value problem

$$Ly = ay'(x) + by'(1-x) = f(x), x \in (0,1), \quad (1.1)$$

$$\alpha y(0) + \beta y(1) = 0, \tag{1.2}$$

in the space  $L^2(0,1)$ , where  $a, b, \alpha, \beta$  are arbitrary complex constants, satisfying the condition

$$(|a| + |b|)(|\alpha| + |\beta|) \neq 0, b \neq 0 \tag{2.1}$$

$f(x)$  is a continuous function on the segment  $[0,1]$ ,  $y(x)$  is a unknown, continuously differentiable function.

**Definition 2.1.** If the boundary value problem

$$Ly = ay'(x) + by'(1-x) - \lambda y(x) = 0, \tag{1.1}$$

$$\alpha y(0) + \beta y(1) = 0, \tag{1.2}$$

has only trivial solution at any complex  $\lambda$ , then it is called Volterra.

**Theorem 2.1.** Boundary value problem (1.1) - (1.2) is Volterra if and only if

1)  $(a^2 - b^2)(\alpha^2 - \beta^2) \neq 0$ ; (2.2)

2)  $b(\alpha^2 + \beta^2) + 2\alpha\beta a = 0$ . (2.3)

**Proof.**

**a) Necessity.** Let boundary value problem (1.1) - (1.2) be Volterra, then  $(a^2 - b^2)(\alpha + \beta) \neq 0$ , otherwise  $\lambda_0 = 0$  is eigenvalue, which contradicts the Volterra property of the boundary value problem (1.1) - (1.2). If the condition  $(a^2 - b^2)(\alpha + \beta) \neq 0$  holds, then there exists inverse operator  $L^{-1}$ , which has the form

$$L^{-1}f(x) = \int_0^1 K(x,t)f(t)dt,$$

where

$$K(x,t) = \frac{a[\alpha\theta(x-t) - \beta\theta(t-x)] - b[\alpha\theta(x-1+t) - \beta\theta(1-t-x)]}{(a^2 - b^2)(\alpha + \beta)}.$$

If the operator  $L$  does not any nonzero eigenvalues, then the operator  $L^{-1}$  also does not have nonzero eigenvalues, consequently, kernel operator  $(L^{-1})^2$  does not have nonzero eigenvalues. Then, by the Lidsky theorem [59], we get

$$SpL^{-2} = 0.$$

Now we calculate the left part of this formula

$$L^{-2}f(x) = \int_0^1 K^2(x,t)f(t)dt,$$

where

$$K^2(x,t) = \int_0^1 K(x,\xi)K(\xi,t)d\xi.$$

It is known that (by the Gaal theorem) [60]

$$SpL^{-2} = \int_0^1 K^2(x,x) dx,$$

thus

$$SpL^{-2} = \int_0^1 \int_0^1 K(x,\xi)K(\xi,t)d\xi dx.$$

It remains to calculate this double integral:

$$K(x,\xi)K(\xi,t) = ?$$

$$K(x, \xi) = \frac{a[\alpha\theta(x - \xi) - \beta\theta(\xi - x)] - b[\alpha\theta(x - 1 + \xi) - \beta\theta(1 - \xi - x)]}{(a^2 - b^2)(\alpha + \beta)} = \frac{K[1] - K[2]}{(a^2 - b^2)(\alpha + \beta)};$$

$$K(\xi, x) = \frac{a[\alpha\theta(\xi - x) - \beta\theta(x - \xi)] - b[\alpha\theta(\xi - 1 + x) - \beta\theta(1 - x - \xi)]}{(a^2 - b^2)(\alpha + \beta)} = \frac{K[3] - K[2]}{(a^2 - b^2)(\alpha + \beta)}.$$

$$K[1] \cdot K[3] = -\alpha\beta\theta(x - \xi) - \beta\alpha\theta(\xi - x) = -\alpha\beta[\theta(x - \xi) + \theta(\xi - x)] = -\alpha\beta;$$

$$K[1] \cdot K[2] = \alpha^2\theta(x - \xi) \cdot \theta(x + \xi - 1) - \alpha\beta\theta(x - \xi)\theta(1 - x - \xi) - \\ - \beta\alpha\theta(\xi - x)\theta(x + \xi - 1) + \beta^2\theta(\xi - x) \cdot \theta(1 - x - \xi);$$

$$K[2] \cdot K[3] = \alpha^2\theta(x + \xi - 1) \cdot \theta(\xi - x) - \alpha\beta\theta(x + \xi - 1)\theta(x - \xi) - \\ - \beta\alpha\theta(1 - x - \xi)\theta(\xi - x) + \beta^2\theta(1 - x - \xi) \cdot \theta(x - \xi);$$

$$K[2] \cdot K[2] = \alpha^2\theta(x + \xi - 1) + \beta^2\theta(1 - x - \xi);$$

Consequently,

$$K(x, \xi) = \frac{aK[1] - bK[2]}{(a^2 - b^2)(\alpha + \beta)}, K(\xi, x) = \frac{aK[3] - bK[2]}{(a^2 - b^2)(\alpha + \beta)},$$

$$K(x, \xi) \cdot K(\xi, x) = \frac{a^2K[1] \cdot K[3] - abK[1] \cdot K[2] - baK[2] \cdot K[3] + b^2K^2[2]}{(a^2 - b^2)(\alpha + \beta)} =$$

$$= a^2(-\alpha\beta) - ab[\alpha^2\theta(x - \xi) \cdot \theta(x + \xi - 1) - \alpha\beta\theta(x - \xi)\theta(1 - x - \xi) -$$

$$- \beta\alpha\theta(\xi - x)\theta(x + \xi - 1) + \beta^2\theta(\xi - x) \cdot \theta(1 - x - \xi)] -$$

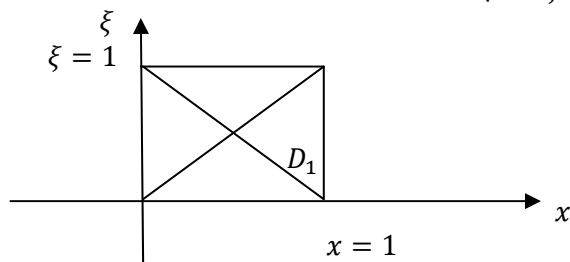
$$- ba[\alpha^2\theta(x + \xi - 1) \cdot \theta(\xi - x) - \alpha\beta\theta(x + \xi - 1)\theta(x - \xi) -$$

$$- \beta\alpha\theta(1 - x - \xi)\theta(\xi - x) + \beta^2\theta(1 - x - \xi) \cdot \theta(x - \xi)] +$$

$$+ b^2[\alpha^2\theta(x + \xi - 1) + \beta^2\theta(1 - x - \xi)] / (a^2 - b^2)(\alpha + \beta)^2;$$

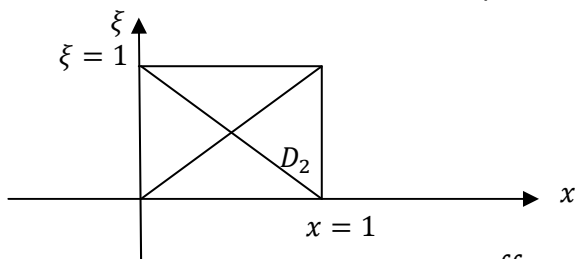
We calculate the following integrals:

$$1) \int_0^1 \int_0^1 \mathbb{H}(x - \xi)\theta(x + \xi - 1) d\xi dx = \left| \begin{array}{l} x - \xi > 0, \\ x + \xi - 1 > 0, \end{array} \Rightarrow x > \xi > 1 - x \right|$$



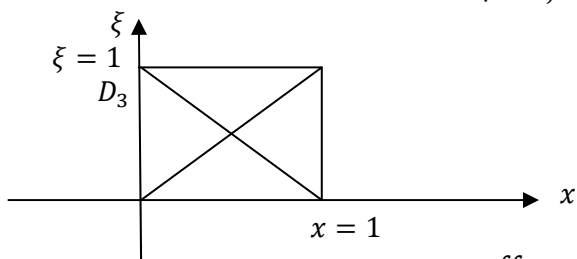
$$= \iint_{D_1} d\xi dx = \frac{1}{4};$$

$$2) \int_0^1 \int_0^1 \theta(x - \xi)\theta(1 - x - \xi)d\xi dx = \left| \begin{array}{l} x - \xi > 0, \\ 1 - x - \xi > 0, \end{array} \Rightarrow \begin{array}{l} x > \xi, \\ 1 - \xi > x > \xi, \end{array} \right|$$



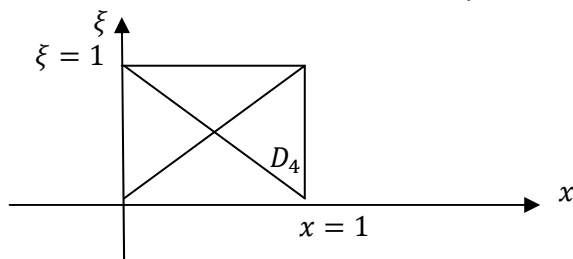
$$= \iint_{D_2} d\xi dx = \frac{1}{4};$$

$$3) \int_0^1 \int_0^1 \theta(\xi - x)\theta(x + \xi - 1)d\xi dx = \left| \begin{array}{l} \xi - x > 0, \\ x + \xi - 1 > 0, \end{array} \Rightarrow \begin{array}{l} \xi > x, \\ \xi > 1 - x, \end{array} \right|$$



$$= \iint_{D_3} d\xi dx = \frac{1}{4};$$

$$4) \int_0^1 \int_0^1 \theta(\xi - x)\theta(1 - x - \xi)d\xi dx = \left| \begin{array}{l} x - \xi > 0, \\ 1 - x - \xi > 0, \end{array} \right|$$



$$= \iint_{D_4} d\xi dx = \frac{1}{4};$$

$$5) \int_0^1 \int_0^1 [\alpha^2 \theta(x + \xi - 1) + \beta^2 \theta(1 - x - \xi)] d\xi dx = ?$$

$$\begin{aligned} \int_0^1 \int_0^1 \alpha^2 \theta(x + \xi - 1) d\xi dx &= \int_0^1 dx \int_{1-x}^x \alpha^2 d\xi = \int_0^1 (1 - 1 + x) \alpha^2 dx = \\ &= \alpha^2 \int_0^1 x dx = \alpha^2 \frac{x^2}{2} \Big|_0^1 = \frac{\alpha^2}{2}; \end{aligned}$$

$$\begin{aligned} \int_0^1 \int_0^1 \beta^2 \theta(1 - x - \xi) dx d\xi &= \beta^2 \int_0^1 dx \int_0^{1-x} d\xi = \beta^2 \int_0^1 (1 - x) dx = \\ &= \beta^2 \left( x - \frac{x^2}{2} \right) \Big|_0^1 = \beta^2 \left( 1 - \frac{1}{2} \right) = \frac{\beta^2}{2}; \end{aligned}$$

$$\begin{aligned}
SpL^{-2} &= \int_0^1 \int_0^1 K(x, \xi)K(\xi, t)d\xi dx = \\
&= \frac{a^2(-\alpha\beta) - \frac{ab}{4}(\alpha^2 - \alpha\beta - \beta\alpha + \beta^2) - \frac{ba}{4}(\alpha^2 - \alpha\beta - \beta\alpha + \beta^2) + b^2 \frac{\alpha^2 + \beta^2}{2}}{(a^2 - b^2)^2(\alpha + \beta)^2} \\
&= \frac{-2\alpha\beta a^2 - ab(\alpha - \beta)^2 + b^2(\alpha^2 + \beta^2)}{2(a^2 - b^2)^2(\alpha + \beta)^2} = \\
&= \frac{-2\alpha\beta a^2 - ab\alpha^2 + 2\alpha\beta ab - ab\beta^2 + b^2\alpha^2 + b^2\beta^2}{2(a^2 - b^2)^2(\alpha + \beta)^2} = \\
&= \frac{[\alpha^2(b^2 - ab) + \beta^2(b^2 - ab) + 2\alpha\beta ab - 2\alpha\beta a^2]}{2(a^2 - b^2)^2(\alpha + \beta)^2} = \\
&= \frac{(b^2 - ab)(\alpha^2 + \beta^2) + 2\alpha\beta(ab - a^2)}{2(a^2 - b^2)^2(\alpha + \beta)^2} = \\
&= \frac{b(b - a)(\alpha^2 + \beta^2) + 2\alpha\beta a(b - a)}{2(a^2 - b^2)^2(\alpha + \beta)^2} = \\
&= \frac{(b - a)[b(\alpha^2 + \beta^2) + 2\alpha\beta a]}{2(a^2 - b^2)^2(\alpha + \beta)^2} = 0.
\end{aligned}$$

Since  $b - a \neq 0$ , then  $b(\alpha^2 + \beta^2) + 2\alpha\beta a = 0$ , thus the necessity of the second condition is proved.

If  $\beta = -\alpha$ , then  $b \cdot 2\alpha^2 - 2\alpha^2 \cdot a = 2\alpha^2(b - a) = 0$ , since  $\alpha \neq 0$ , therefore  $b - a = 0$ , it is impossible, thus  $\beta \neq -\alpha$  or  $\beta + \alpha \neq 0$ .

Similarly, if  $\beta = \alpha$ , then  $b \cdot 2\alpha^2 + 2\alpha^2 \cdot a = 0$ ,  $2\alpha^2(b + a) = 0$ , since  $\alpha \neq 0$ , then  $b + a = 0$ , it is impossible, therefore  $\beta - \alpha \neq 0$ . Earlier we showed that  $(a^2 - b^2)(\alpha + \beta) \neq 0$ , due to the last condition, we have  $(a^2 - b^2)(\alpha^2 - \beta^2) \neq 0$ , t.e. we get the inequality (4.2).

#### 6) Sufficiency.

Let  $(a^2 - b^2)(\alpha^2 - \beta^2) \neq 0$  and  $b(\alpha^2 + \beta^2) + 2\alpha\beta a = 0$ . From the equation (1.1) we have

$$\begin{aligned}
T \frac{d}{dx} y(x) &= \lambda y(x), y'(x) = \lambda T^{-1} y(x), \\
y'(x) &= \lambda \frac{aI - bS}{a^2 - b^2} y(x).
\end{aligned}$$

Differentiating this equation, we obtain

$$\begin{aligned}
y''(x) &= \lambda \frac{ay' + by'(1-x)}{a^2 - b^2} = \lambda \frac{aI + bS}{a^2 - b^2} y'(x) = \\
&= \lambda^2 \frac{aI + bS}{a^2 - b^2} \cdot \frac{aI - bS}{a^2 - b^2} = \frac{\lambda^2}{a^2 - b^2} y(x).
\end{aligned}$$

**Lemma 2.4.** If  $\lambda \neq 0$  is a eigenvalue of the boundary value problem

$$ay'(x) + by'(1-x) = \lambda y(x), x \in (0,1), \quad (1.1)$$

$$\alpha y(0) + \beta y(1) = 0, \quad (1.2)$$

where  $b \neq 0$ ,  $|\alpha| + |\beta| \neq 0$ ,  $a^2 - b^2 \neq 0$ , then the quantity

$$\mu^2 = \frac{\lambda^2}{a^2 - b^2} \quad (2.4)$$

is eigenvalue of the Sturm - Liouville operator

$$y''(x) = \mu^2 y(x), x \in (0,1), \quad (2.5)$$

$$\begin{cases} \alpha y(0) + \beta y(1) = 0, \\ \alpha[ay'(0) + by'(1)] + \beta[ay'(1) + by'(0)] = 0, \end{cases} \quad (2.6)$$

**Corollary 2.1.** If the Sturm - Liouville problem (2.5)-(2.6) does not have nonzero eigenvalues, then the problem (1.1)-(1.2) also does not them.

Boundary value problem (2.5)-(2.6) does not eigenvalues if and only if

$$\Delta_{24} = 0, \Delta_{14} + \Delta_{32} = 0, \Delta_{13} = 0,$$

where  $\Delta_{ij}$  are minors of the matrix

$$\begin{pmatrix} \alpha & 0 & \beta & 0 \\ 0 & \alpha a + \beta b & 0 & \alpha b + \beta a \end{pmatrix}.$$

It is obvious that

$$\Delta_{24} = 0, \Delta_{14} = \alpha(\alpha b + \beta a), \Delta_{23} = -\beta(\alpha a + \beta b), \Delta_{13} = 0.$$

Moreover,

$$\begin{aligned} \Delta_{14} + \Delta_{32} &= \alpha(\alpha b + \beta a) + \beta(\alpha a + \beta b) = \\ &= \alpha^2 b + \alpha\beta a + \beta\alpha a + \beta^2 b = b(\alpha^2 + \beta^2) + 2\alpha\beta a = 0. \end{aligned}$$

Consequently, the problem (4.5) - (4.6) is Volterra, i.e. does not have eigenvalues on all complex plane, thus the problem (1.1) - (1.2) also does not have nonzero eigenvalues.

**Remark 2.1.**

**Other proof of sufficiency.**

If  $\alpha^2 - \beta^2 \neq 0$  and  $a^2 - b^2 \neq 0$ , then our boundary value problem is equivalent to the generalized Cauchy problem

$$\begin{cases} z'(x) = \lambda(\alpha^2 - \beta^2)Q^{-1}P^{-1}Q^{-1}z(x), \\ z(0) = 0; \end{cases} \quad (2.7), (2.8)$$

where

$$Pu(x) = au(x) + bu(1-x) = (aI + bS)u(x),$$

$$Qu(x) = \alpha u(x) + \beta u(1-x) = (\alpha I + \beta S)u(x),$$

$$Q^{-1} = \frac{\alpha I - \beta S}{\alpha^2 - \beta^2}, P^{-1} = \frac{\alpha I - bS}{\alpha^2 - b^2},$$

$$P^{-1}Q^{-1} = \frac{\alpha I - bS}{\alpha^2 - b^2} \cdot \frac{\alpha I - \beta S}{\alpha^2 - \beta^2} = \frac{\alpha\alpha I - a\beta S - b\alpha S + b\beta S}{(\alpha^2 - b^2)(\alpha^2 - \beta^2)} =$$

$$= \frac{\alpha\alpha I - (a\beta + b\alpha)S + b\beta S}{(\alpha^2 - b^2)(\alpha^2 - \beta^2)} = \frac{(\alpha\alpha + b\beta)I - (a\beta + b\alpha)S}{(\alpha^2 - b^2)(\alpha^2 - \beta^2)};$$

$$Q^{-1}P^{-1}Q^{-1} = \frac{\alpha I - \beta S}{\alpha^2 - \beta^2} \cdot \frac{(\alpha\alpha + b\beta)I - (a\beta + b\alpha)S}{(\alpha^2 - b^2)(\alpha^2 - \beta^2)} =$$

$$= \frac{\alpha(\alpha\alpha + b\beta)I + \beta(a\beta + b\alpha)I - \alpha(a\beta + b\alpha)S - \beta(\alpha\alpha + b\beta)S}{(\alpha^2 - b^2)(\alpha^2 - \beta^2)^2} =$$

$$= \frac{(\alpha^2 a + \alpha\beta b + \beta^2 a + \alpha\beta b)I - (a\alpha\beta + \alpha^2 b + \beta\alpha a + \beta^2 b)S}{(\alpha^2 - b^2)(\alpha^2 - \beta^2)^2} =$$

$$= \frac{[a(\alpha^2 + \beta^2) + 2\alpha\beta b]I - [b(\alpha^2 + \beta^2) + 2\alpha\beta a]S}{(\alpha^2 - b^2)(\alpha^2 - \beta^2)^2}.$$

Consequently, Cauchy problem (2.7) - (2.8) takes the following form

$$\begin{cases} z'(x) = \lambda \frac{[\alpha(\alpha^2 + \beta^2) + 2\alpha\beta b]}{(\alpha^2 - b^2)(\alpha^2 - \beta^2)} z(x), \\ z(0) = 0; \end{cases}$$

It is obvious that this Cauchy problem has only trivial solution at any value of  $\lambda$ .

**Remark 2.2.** This proof prompted us that  $\alpha^2 - \beta^2 \neq 0$ . The fact is that we did not check the conditions

- 1)  $(\alpha^2 - b^2)(\alpha + \beta) \neq 0$ ,
  - 2)  $b(\alpha^2 + \beta^2) + 2\alpha\beta a = 0$
- on compatibility.

From the second condition when  $\beta = \alpha$ , we have

$$b \cdot 2\alpha^2 + 2\alpha^2 \cdot a = 0, 2\alpha^2(b + a) = 0.$$

If  $\alpha = 0$ , then  $\beta = 0$ , which is not acceptable therefore  $b + a = 0$ , and this contradicts the first condition, therefore  $\alpha \neq 0$  or  $\beta - \alpha \neq 0$ . And the final look of the Volterra is

- 1)  $(\alpha^2 - b^2)(\alpha^2 - \beta^2) \neq 0$ ,
- 2)  $b(\alpha^2 + \beta^2) + 2\alpha\beta a = 0$ .

**2.4. On basis property.**

**Lemma 2.5.** If

- a)  $(\alpha^2 - b^2)(\alpha^2 - \beta^2) \neq 0$ ,
- б) function  $y(x, \pm\lambda)$  is eigenfunction for the boundary value problem

$$ay'(x) + by'(1 - x) = \pm\lambda y(x), x \in (0,1), \tag{2.9}$$

$$\alpha y(0) + \beta y(1) = 0, \tag{2.10}$$

where  $|\alpha| + |\beta| \neq 0$ , then it is eigenfunction for the Sturm - Liouville boundary value problem

$$y''(x) = \mu^2 y(x), x \in (0,1), \tag{2.11}$$

$$\begin{cases} \alpha y(0) + \beta y(1) = 0, \\ ((\alpha\alpha + b\beta)y'(0) + (a\beta + b\alpha)y'(1) = 0, \end{cases} \tag{2.12}$$

where

$$\mu^2 = \frac{\lambda^2}{b^2 - a^2}. \tag{2.13}$$

Inversely, if the quantity  $\mu^2 = \frac{\lambda^2}{b^2 - a^2}$  is eigenvalue of the Sturm - Liouville boundary value problem

$$y''(x) = \mu^2 y(x), x \in (0,1), \tag{2.11}$$

$$\begin{cases} \alpha y(0) + \beta y(1) = 0, \\ ((\alpha\alpha + b\beta)y'(0) + (a\beta + b\alpha)y'(1) = 0, \end{cases} \tag{2.12}$$

Then the function  $y(x)$  is eigenfunction for the boundary value problem

$$ay'(x) + by'(1 - x) = \pm\lambda y(x), \tag{2.9}$$

$$\alpha y(0) + \beta y(1) = 0, \tag{2.10}$$

**Proof.** Suppose that (2.9) - (2.10) holds. Then, assuming

$$\begin{aligned} Tu(x) &= au(x) + bu(1 - x) = (aI + bS)u(x), \\ Su(x) &= u(1 - x) \end{aligned}$$

we have

$$\begin{aligned} Ty'(x) &= \pm\lambda y(x), \\ y'(x) &= \pm\lambda T^{-1}y(x) = \pm\lambda \frac{aI - bS}{a^2 - b^2} y(x), \\ y''(x) &= \pm\lambda \frac{ay'(x) + by'(1-x)}{a^2 - b^2} = \pm\lambda \frac{aI + bS}{a^2 - b^2} y'(x) = \\ &= \frac{\pm\lambda}{a^2 - b^2} Ty'(x) = \frac{\lambda^2}{a^2 - b^2} y(x), \end{aligned}$$

consequently,

$$-y''(x) = \frac{\lambda^2}{b^2 - a^2} y(x).$$

Further, from the equation (2.9) due to the boundary condition (2.10), we obtain

$$\alpha[ay'(0) + by'(1)] + \beta[ay'(1) + by'(0)] = \pm\lambda[\alpha y(0) + \beta y(1)] = 0.$$

Therefore, the second boundary condition has the form

$$(a\alpha + b\beta)y'(0) + (a\beta + b\alpha)y'(1) = 0.$$

Inversely, let function  $y(x, \mu)$  be a eigenfunction of the Sturm – Liouville boundary value problem

$$y''(x) = \mu^2 y(x), x \in (0,1), \tag{2.11}$$

$$\begin{cases} \alpha y(0) + \beta y(1) = 0, \\ (a\alpha + b\beta)y'(0) + (a\beta + b\alpha)y'(1) = 0, \end{cases} \tag{2.12}$$

where

$$\mu^2 = \frac{\lambda^2}{b^2 - a^2}. \tag{2.13}$$

We find a eigenfunction, corresponding to the eigenvalue  $\mu^2$ .

General solution of the equation (2.11) has the form

$$y(x, \mu) = Ae^{i\mu x} + Be^{-i\mu x},$$

where  $A, B$  are arbitrary constants. Putting this expansion into the boundary condition (2.12), we receive a system of homogeneous equations for unknowns  $A, B$ .

$$\begin{aligned} \alpha y(0) + \beta y(1) &= \alpha(A + B) + \beta(Ae^{i\mu} + Be^{-i\mu}) = \\ &= A(\alpha + \beta e^{i\mu}) + B(\alpha + \beta e^{-i\mu}) = 0; \end{aligned}$$

$$y'(x, \mu) = i\mu(Ae^{i\mu x} - Be^{-i\mu x}), y'(0) = i\mu(A - B),$$

$$y'(1) = i\mu(Ae^{i\mu} - Be^{-i\mu}),$$

$$(a\alpha + b\beta)i\mu(A - B) + (a\beta + b\alpha)i\mu(Ae^{i\mu} - Be^{-i\mu}) = 0, i\mu \neq 0;$$

$$(A - B)(a\alpha + b\beta) + (a\beta + b\alpha)(Ae^{i\mu} - Be^{-i\mu}) = 0,$$

$$A[a\alpha + b\beta + (a\beta + b\alpha)e^{i\mu}] + B[-a\alpha - b\beta - (a\beta + b\alpha)e^{-i\mu}] = 0,$$

$$\begin{cases} A(\alpha + \beta e^{i\mu}) + B(\alpha + \beta e^{-i\mu}) = 0, \\ A[a\alpha + b\beta + (a\beta + b\alpha)e^{i\mu}] + B[-a\alpha - b\beta - (a\beta + b\alpha)e^{-i\mu}] = 0. \end{cases}$$



We calculate determinant of this system of equations

$$\begin{aligned}\Delta(\mu) &= \begin{vmatrix} \alpha + \beta e^{i\mu} & \alpha + \beta e^{-i\mu} \\ a\alpha + b\beta + (a\beta + b\alpha)e^{i\mu} & -a\alpha - b\beta - (a\beta + b\alpha)e^{-i\mu} \end{vmatrix} = \\ &= -(\alpha + \beta e^{i\mu})[a\alpha + b\beta + (a\beta + b\alpha)e^{-i\mu}] - \\ &-(\alpha + \beta e^{-i\mu})[a\alpha + b\beta + (a\beta + b\alpha)e^{i\mu}] = 0. \quad (2.14)\end{aligned}$$

Transforming this expression, we get

$$\begin{aligned}\Delta(\mu) &= -[\alpha(a\alpha + b\beta) + \alpha(a\beta + b\alpha)e^{-i\mu} + \beta(a\alpha + b\beta)e^{i\mu} + \beta(\frac{a\beta}{\alpha} + b\alpha) + \\ &+ \alpha(a\alpha + b\beta) + \alpha(a\beta + b\alpha)e^{i\mu} + \beta(a\alpha + b\beta)e^{-i\mu} + \beta(a\beta + b\alpha)] = \\ &= -\{2\alpha(a\alpha + b\beta) + 2\beta(a\beta + b\alpha) + [\alpha(a\beta + b\alpha) + \beta(a\alpha + b\beta)]e^{-i\mu} + \\ &+ [\alpha(a\beta + b\alpha) + \beta(a\alpha + b\beta)]e^{i\mu}\} = \\ &= -\{2\alpha^2 a + 2\alpha\beta b + 2\beta^2 a + 2\alpha\beta b + \\ &+ [\alpha(a\beta + b\alpha) + \beta(a\alpha + b\beta)](e^{i\mu} + e^{-i\mu})\} = \\ &= -2[(\alpha^2 + \beta^2)a + 2\alpha\beta b + (\alpha\beta a + b\alpha^2 + \beta\alpha a + b\beta^2) \cos \mu] = \\ &= -2\{a(\alpha^2 + \beta^2) + 2\alpha\beta b + [b(\alpha^2 + \beta^2) + 2\alpha\beta a] \cos \mu\}.\end{aligned}$$

**Remark 2.3.** If

- 1)  $b(\alpha^2 + \beta^2) + 2\alpha\beta a = 0$ ;
- 2)  $a(\alpha^2 + \beta^2) + 2\alpha\beta b \neq 0$

then  $\Delta(\mu) \neq 0$ , i.e. the problem is Volterra.

In this case:

$$\begin{aligned}2) - 1) &= (b^2 + \beta^2)(a - b) + 2\alpha\beta(b - a) = (a - b)(\alpha - \beta)^2 \neq 0, \\ 2) + 1) &= (\alpha^2 + \beta^2)(a + b) + 2\alpha\beta(b + a) = (a + b)(\alpha + \beta)^2 \neq 0.\end{aligned}$$

Therefore,  $(a^2 - b^2)(\alpha^2 - \beta^2)^2 \neq 0$ , that coincides with the first condition of Volterra property, see (2.2).

Supposing  $b(b^2 + \beta^2) + 2\alpha\beta a \neq 0$ , from the equation  $\Delta(\mu) = 0$  we have

$$\begin{aligned}\cos \mu &= -\frac{a(\alpha^2 + \beta^2) + 2\alpha\beta b}{b(\alpha^2 + \beta^2) + 2\alpha\beta a}, \\ \mu &= \pm \arccos \left[ -\frac{a(\alpha^2 + \beta^2) + 2\alpha\beta b}{b(\alpha^2 + \beta^2) + 2\alpha\beta a} \right] + 2n\pi, n = 0, \pm 1, \pm 2, \dots\end{aligned}$$

We investigate multiplicity of eigenvalues

$$\dot{\Delta}(\mu) = 2[b(\alpha^2 + \beta^2) + 2\alpha\beta a] \sin \mu.$$

If  $\dot{\Delta}(\mu) = 0$ , then  $\sin \mu = 0$ ,  $\cos \mu = \pm 1$ . Thus

$$-\frac{a(\alpha^2 + \beta^2) + 2\alpha\beta b}{b(\alpha^2 + \beta^2) + 2\alpha\beta a} = \pm 1.$$

a) If

$a(\alpha^2 + \beta^2) + 2\alpha\beta b = b(\alpha^2 + \beta^2) + 2\alpha\beta a$ , then

$$\begin{aligned}(a - b)(\alpha^2 + \beta^2) + 2\alpha\beta(b - a) &= (a - b)(\alpha^2 + \beta^2 - 2\alpha\beta) = \\ &= (a - b)(\alpha - \beta)^2 = 0;\end{aligned}$$

6) If

$$\begin{aligned} a(\alpha^2 + \beta^2) + 2\alpha\beta b &= -b(\alpha^2 + \beta^2) - 2\alpha\beta a, \\ (\alpha^2 + \beta^2)(a + b) + 2\alpha\beta(b + a) &= (a + b)(\alpha^2 + \beta^2 + 2\alpha\beta) = \\ &= (a + b)(\alpha + \beta)^2 = 0. \end{aligned}$$

In our case,  $(\alpha^2 - b^2)(\alpha^2 - \beta^2) \neq 0$ , therefore  $\Delta(\mu) \neq 0$  and the Sturm - Liouville problem does not have associated functions.

Supposing,

$$A = K(\alpha + \beta e^{-i\mu}), B = (-\alpha - \beta e^{i\mu})K,$$

where  $K$  is arbitrary constant, we obtain (find) eigenfunctions of the Sturm - Liouville boundary value problem

$$y(x, \mu) = K(\alpha + \beta e^{-i\mu})e^{i\mu x} - K(\alpha + \beta e^{i\mu})e^{-i\mu x}.$$

Then

$$\begin{aligned} y'(x, \mu) &= K[i\mu(\alpha + \beta e^{-i\mu})e^{i\mu x} + i\mu(\alpha + \beta e^{i\mu})e^{-i\mu x}], \\ y'(1-x, \mu) &= K[i\mu(\alpha + \beta e^{-i\mu})e^{i\mu(1-x)} + i\mu(\alpha + \beta e^{i\mu})e^{-i\mu(1-x)}] = \\ &= K[i\mu(\alpha e^{i\mu} + \beta)e^{-i\mu x} + i\mu(\alpha e^{-i\mu} + \beta)e^{i\mu x}]; \\ ay' + by'(1-x) &= i\mu K\{[a(\alpha + \beta e^{-i\mu}) + b(\alpha e^{-i\mu} + \beta)]e^{i\mu x} + \\ &+ [a(\alpha + \beta e^{i\mu}) + b(\alpha e^{i\mu} + \beta)]e^{-i\mu x}\}. \end{aligned}$$

Further, from the equality  $\Delta(\mu) = 0$  it follows that rows of this determinant (2.14) are linear dependent, therefore we have

$$\begin{aligned} a\alpha + a\beta e^{-i\mu} + b\alpha e^{-i\mu} + b\beta &= \\ = a\alpha + b\beta + (a\beta + b\alpha)e^{-i\mu} &= -C(\alpha + \beta e^{-i\mu}), \end{aligned} \quad (2.15)$$

$$\begin{aligned} a\alpha + a\beta e^{i\mu} + b\alpha e^{i\mu} + b\beta &= \\ = a\alpha + b\beta + (a\beta + b\alpha)e^{i\mu} &= C(\alpha + \beta e^{i\mu}), \end{aligned} \quad (2.16)$$

Consequently,

$$\begin{aligned} ay' + by'(1-x) &= i\mu K \cdot C[-(\alpha + \beta e^{-i\mu})e^{i\mu x} + (\alpha + \beta e^{i\mu})e^{-i\mu x}] = \\ &= i\mu K \cdot C[(\alpha + \beta e^{-i\mu})e^{i\mu x} - (\alpha + \beta e^{i\mu})e^{-i\mu x}] = \\ &= -i\mu \cdot Cy(x, \mu) = -\frac{\lambda}{\sqrt{a^2 - b^2}} Cy(x, \mu). \end{aligned} \quad (2.17)$$

Calculate the constant  $C$ . If we sum up formulas (2.15) and (2.16), then we get

$$\begin{aligned} 2(a\alpha + b\beta) + (a\beta + b\alpha)(e^{i\mu} + e^{-i\mu}) &= C(e^{i\mu} - e^{-i\mu}) \cdot \beta, \Rightarrow \\ a\alpha + b\beta + (a\beta + b\alpha) \cos \mu &= C \cdot \beta i \sin \mu. \end{aligned}$$

Subtracting the formula (2.15) from (2.16), we have

$$\begin{aligned} (a\beta + b\alpha)(e^{i\mu} - e^{-i\mu}) &= C[2\alpha + \beta(e^{i\mu} + e^{-i\mu})], \Rightarrow \\ i(a\beta + b\alpha) \sin \mu &= C[\alpha + \beta \cos \mu]; \end{aligned}$$

Consequently,

$$C = \frac{i(a\beta + b\alpha) \sin \mu}{\alpha + \beta \cos \mu} = \frac{a\alpha + b\beta + (a\beta + b\alpha) \cos \mu}{i\beta \sin \mu};$$

In our situation

$$\cos \mu = -\frac{a(\alpha^2 + \beta^2) + 2\alpha\beta b}{b(\alpha^2 + \beta^2) + 2\alpha\beta a},$$

Thus

$$\begin{aligned}
 \alpha + \beta \cos \mu &= \alpha - \frac{a\beta(\alpha^2 + \beta^2) + 2\alpha\beta^2b}{b(\alpha^2 + \beta^2) + 2\alpha\beta a} = \\
 &= \frac{ab(\alpha^2 + \beta^2) + 2\alpha^2\beta a - a\beta(\alpha^2 + \beta^2) - 2\alpha\beta^2b}{b(\alpha^2 + \beta^2) + 2\alpha\beta a} = \\
 &= \frac{(\alpha^2 + \beta^2)(ab - a\beta) + 2\alpha\beta(\alpha a - \beta b)}{b(\alpha^2 + \beta^2) + 2\alpha\beta a} = \\
 &= \frac{b[\alpha(\alpha^2 + \beta^2) - 2\alpha\beta^2] + a[-\beta(\alpha^2 + \beta^2) + 2\alpha^2\beta]}{b(\alpha^2 + \beta^2) + 2\alpha\beta a} = \\
 &= \frac{b\alpha(\alpha^2 - \beta^2) + a\beta(\alpha^2 - \beta^2)}{b(\alpha^2 + \beta^2) + 2\alpha\beta a} = \frac{(\alpha^2 - \beta^2)(b\alpha + a\beta)}{b(\alpha^2 + \beta^2) + 2\alpha\beta a}, \\
 \sin \mu &= \pm \sqrt{1 - \frac{[a(\alpha^2 + \beta^2) + 2\alpha\beta b]^2}{[b(\alpha^2 + \beta^2) + 2\alpha\beta a]^2}} = \frac{\pm\sqrt{b^2 - a^2}(\alpha^2 - \beta^2)}{b(\alpha^2 + \beta^2) + 2\alpha\beta a}, \\
 C &= \frac{\pm i(a\beta + b\alpha)\sqrt{b^2 - a^2}(\alpha^2 - \beta^2)}{b(\alpha^2 + \beta^2) + 2\alpha\beta a} = \frac{(\alpha^2 - \beta^2)(b\alpha + a\beta)}{b(\alpha^2 + \beta^2) + 2\alpha\beta a} = \\
 &= \pm i\sqrt{b^2 - a^2} = \pm\sqrt{a^2 - b^2}.
 \end{aligned}$$

Putting this expression into the formula (2.17), we have

$$ay'(x) + by'(1-x) = \mp \lambda y(x, \mu),$$

which was required to prove.

**Problem.** Prove that the system of eigenfunctions of the Sturm - Liouville boundary value problem is basis

$$-y''(x) = \mu^2 y(x), x \in (0,1), \quad (2.11)$$

$$\begin{cases} \alpha y(0) + \beta y(1) = 0, \\ (\alpha\alpha + b\beta)y'(0) + (a\beta + b\alpha)y'(1) = 0, \end{cases} \quad (2.12)$$

where  $(\alpha^2 - \beta^2)(\alpha^2 - b^2) \neq 0$ , by the Kesselman - Mikhailov Test [57-58].

**Solution.** Boundary matrix of the boundary value problem (2.11) - (2.12) has the form

$$\begin{pmatrix} \alpha & 0 & \beta & 0 \\ 0 & \alpha\alpha + b\beta & 0 & a\beta + b\alpha \end{pmatrix}.$$

We calculate minors of this matrix

$$\begin{aligned}
 \Delta_{12} &= \alpha(\alpha\alpha + b\beta), \Delta_{13} = 0, \Delta_{14} = \alpha(a\beta + b\alpha), \\
 \Delta_{23} &= -\beta(\alpha\alpha + b\beta), \Delta_{24} = 0, \Delta_{34} = \beta(a\beta + b\alpha);
 \end{aligned}$$

Hence,

$$\begin{aligned}
 \Delta_{14} + \Delta_{32} &= \alpha(a\beta + b\alpha) + \beta(\alpha\alpha + b\beta) = \\
 &= b(\alpha^2 + \beta^2) + 2\alpha\beta a \neq 0,
 \end{aligned}$$

otherwise the problem is Volterra.

First we check the Birkhoff regularity condition [54]; for this, we rearrange rows of the boundary matrix

$$\begin{pmatrix} a_0 & a_1 & b_0 & b_1 \\ 0 & \alpha a + b\beta & 0 & a\beta + b\alpha \\ \alpha & 0 & \beta & 0 \\ c_0 & c_1 & d_0 & d_1 \end{pmatrix}$$

1)  $a_1 d_1 - b_1 c_1 = 0$ ,

2)  $a_1 d_1 - b_1 c_1 = 0, |a_1| + |b_1| = |\alpha a + b\beta| + |a\beta + b\alpha| > 0$ ,

$$b_1 c_0 + a_1 d_0 = \alpha(a\beta + b\alpha) + \beta(\alpha a + \beta b) = b(\alpha^2 + \beta^2) + 2\alpha\beta a \neq 0,$$

3)  $a_1 = b_1 = c_1 = d_1 = 0, a_0 d_0 - b_0 c_0 \neq 0$ .

It is obvious that in our case conditions 1) and 3) coincide, only condition 2) remains.

If  $|a_1| + |b_1| = 0$ , then  $a_1 = 0$  and  $b_1 = 0$ . Then we get

$$\begin{cases} \alpha a + b\beta = 0, \\ a\beta + b\alpha = 0. \end{cases}$$

Since  $|a| + |b| \neq 0$ , then  $\alpha^2 - \beta^2 = 0$ ; similarly from the condition  $|a| + |\beta| \neq 0$  we have that  $\alpha^2 - b^2 = 0$ , which is not possible by our condition. Consequently, our boundary value problems (2.12) are regular by the second part of the Birkhoff condition [56].

The Kesselman condition [57] is

$$\Delta_{14}^2 + \Delta_{32}^2 - \Delta_{12}^2 - \Delta_{34}^2 \neq 0.$$

In our case,

$$\begin{aligned} \Delta_{14}^2 + \Delta_{32}^2 &= \alpha^2(a\beta + b\alpha)^2 + \beta^2(\alpha a + \beta b)^2, \\ \Delta_{12}^2 &= \alpha^2(\alpha a + \beta b)^2, \Delta_{34}^2 = \beta^2(a\beta + b\alpha)^2, \Rightarrow \\ \Delta_{14}^2 + \Delta_{32}^2 - \Delta_{12}^2 - \Delta_{34}^2 &= (\alpha^2 - \beta^2)(a\beta + b\alpha)^2 + (\beta^2 - \alpha^2)(\alpha a + \beta b)^2 = \\ &= (\alpha^2 - \beta^2)[(a\beta + b\alpha)^2 - (\alpha a + \beta b)^2] = \\ &= (\alpha^2 - \beta^2)[(a\beta + b\alpha - \alpha a - \beta b)(a\beta + b\alpha + \alpha a + \beta b)] = \\ &= (\alpha^2 - \beta^2)[a(\beta - \alpha) + b(\alpha - \beta)][\beta(a + b) + \alpha(a + b)] = \\ &= (\alpha^2 - \beta^2)(\alpha - \beta)(b - a)(a + b)(\alpha + \beta) = \\ &= (\alpha^2 - \beta^2)(\alpha^2 - \beta^2)(b^2 - a^2) = (\alpha^2 - \beta^2)^2(b^2 - a^2) \neq 0. \end{aligned}$$

Consequently, the eigenfunctions of the Sturm – Liouville boundary value problem (2.11) - (2.12) form a Riesz basis in the space  $L^2(0,1)$ .

**Theorem 2.2.**If

a)  $(\alpha^2 - \beta^2)(a^2 - b^2) \neq 0$ ,

b)  $b(\alpha^2 + \beta^2) + 2\alpha\beta a \neq 0$ ,

then eigenfunctions of the boundary value problem

$$ay'(x) + by'(1-x) = \lambda y(x), x \in (0,1), \tag{1.1}$$

$$\alpha y(0) + \beta y(1) = 0, \tag{1.2}$$

form a Riesz basis in the space  $L^2(0,1)$ .

**Remark 2.5.**

If  $a = 0, b = 1$ , then the boundary value problem (1.1)-(1.2) takes the form

$$y'(1-x) = \lambda y(x), x \in (0,1),$$

$$\alpha y(0) + \beta y(1) = 0.$$

Thus the conditions a) and b) of Theorem 2.2. are transformed as follows:

$$a) \alpha^2 - \beta^2 \neq 0,$$

$$b) \alpha^2 + \beta^2 \neq 0$$

or  $(\alpha^2 - \beta^2)(\alpha^2 + \beta^2) \neq 0$ , or  $\alpha^4 - \beta^4 \neq 0$ , that coincides with the results of [28].

### 3. Research Results

We consider in the space  $L^2(0,1)$  the following boundary value problem

$$Ly = ay'(x) + by'(1-x) = \lambda y(x), x \in (0,1), \quad (1.1)$$

$$\alpha y(0) + \beta y(1) = 0, \quad (1.2)$$

where  $a, b, \alpha, \beta$  are arbitrary, previously known constants, and  $\lambda$  is a spectral parameter,  $y(x)$  is a desired function from the class  $C^1(0,1) \cap C[0,1]$ , and we formulate the obtained results.

**Theorem 3.1.** The boundary value problem (1.1) - (1.2) is Volterra if and only if

$$1) (\alpha^2 - b^2)(\alpha^2 - \beta^2) \neq 0; \quad (2.2)$$

$$2) b(\alpha^2 + \beta^2) + 2\alpha\beta a = 0. \quad (2.3)$$

**Theorem 3.2.** If

$$a) (\alpha^2 - \beta^2)(\alpha^2 - b^2) \neq 0,$$

$$b) b(\alpha^2 + \beta^2) + 2\alpha\beta a \neq 0;$$

then eigenfunctions of the boundary value problem

$$ay'(x) + by'(1-x) = \lambda y(x), x \in (0,1), \quad (1.1)$$

$$\alpha y(0) + \beta y(1) = 0, \quad (1.2)$$

form a Riesz basis in the space  $L^2(0,1)$ .

### 4. Discussion of Results.

**Remark 4.1.** If

$$\begin{cases} b(\alpha^2 + \beta^2) + 2\alpha\beta a = 0, \\ b2\alpha\beta + (\alpha^2 + \beta^2)a = 0, \end{cases}$$

then  $(\alpha^2 + \beta^2)^2 - 4\alpha^2\beta^2 = 0$ , i.e.

$$\begin{aligned} (\alpha^2 + \beta^2)^2 - 4\alpha^2\beta^2 &= (\alpha^2 + \beta^2 - 2\alpha\beta)(\alpha^2 + \beta^2 + 2\alpha\beta) = (\alpha - \beta)^2(\alpha + \beta)^2 \\ &= (\alpha^2 - \beta^2)^2 = 0. \end{aligned}$$

**Remark 4.2.**

a) If  $C = -\sqrt{a^2 - b^2}$ , then from the formula (2.15) we have

$$\begin{aligned} a\alpha + b\beta + (a\beta + b\alpha)e^{-i\mu} &= \sqrt{a^2 - b^2} \cdot (\alpha + \beta e^{-i\mu}), \\ a\alpha + b\beta + (a\beta + b\alpha)e^{-i\mu} &= \beta\sqrt{a^2 - b^2}e^{-i\mu} + \alpha\sqrt{a^2 - b^2}, \\ (a\beta + b\alpha - \beta\sqrt{a^2 - b^2})e^{-i\mu} &= \alpha\sqrt{a^2 - b^2} - a\alpha - b\beta, \\ e^{-i\mu} &= \frac{\alpha\sqrt{a^2 - b^2} - a\alpha - b\beta}{a\beta + b\alpha - \beta\sqrt{a^2 - b^2}} \end{aligned}$$

and from the formula (2.16) we have

$$e^{i\mu} = \frac{-\alpha\sqrt{a^2 - b^2} - a\alpha - b\beta}{a\beta + b\alpha + \beta\sqrt{a^2 - b^2}}, \Rightarrow$$

consequently,  $e^{-i\mu} \cdot e^{i\mu} = 1$ .

б) If  $C = \sqrt{a^2 - b^2}$ , then from the formula (2.15) we have

$$\begin{aligned} a\alpha + b\beta + (a\beta + b\alpha)e^{-i\mu} &= -\sqrt{a^2 - b^2} \cdot (\alpha + \beta e^{-i\mu}), \\ a\alpha + b\beta + (a\beta + b\alpha)e^{-i\mu} &= -\beta\sqrt{a^2 - b^2}e^{-i\mu} - \alpha\sqrt{a^2 - b^2}, \\ (a\beta + b\alpha + \beta\sqrt{a^2 - b^2})e^{-i\mu} &= -\alpha\sqrt{a^2 - b^2} - a\alpha - b\beta, \\ e^{-i\mu} &= -\frac{\alpha\sqrt{a^2 - b^2} + a\alpha + b\beta}{a\beta + b\alpha + \beta\sqrt{a^2 - b^2}} \end{aligned}$$

**5. Findings.** Operator corresponding to this boundary value problem is not semi-bounded; therefore, variational methods are not suitable to study such problems, and this is a distinctive feature of this problem. In our opinion, such operators can be used to construct non-local transformation operators, and apply them to study spectral properties of operator sheaves.

ӘОЖ 517.929

А.Ш.Шалданбаев<sup>1</sup>, С.М.Шаленова<sup>2</sup>, М.Б.Иванова<sup>3</sup>, А.А.Шалданбаева<sup>4</sup>

<sup>1</sup>Халықаралық Silkway университеті, Шымкент қ., Қазақстан;

<sup>2</sup>Ш. Есенов атындағы Каспий мемлекеттік технологиялар мен инжиниринг университеті, Ақтау қ., Қазақстан;

<sup>3</sup>Оңтүстік Қазақстан медициналық академиясы, Шымкент қ., Қазақстан;

<sup>4</sup>Аймақтық әлеуметтік-инновациялық университеті, Шымкент қ., Қазақстан;

#### АРГУМЕНТІ АУЫТҚЫҒАН БІРІНШІ РЕТТІ ДИФФЕРЕНЦИАЛДЫҚ ТЕНДЕУДІҢ ШЕКАРАЛЫҚ ЕСЕБІНІҢ СПЕКТРӘЛДІК ҚАСИЕТТЕРІ ТУРАЛЫ

**Аннотация.** Бұл еңбекте Аргументі ауытқыған бірінші ретті дифференциалдық тендеудің шекаралық есебінің спектрәлдік қасиеттері зерттелді. Бір айта кетері, ауытқу тендеудің үлкен ретті туындыларында кездеседі және оны қосымша шарттарсыз төменгі ретті мүшелерге ауыстыруға болмайды. Спектрәлді қасиеттер ретінде, біз оның меншікті векторлары мен оларға еншілес векторлар системасының толымдылығы мен базистігін таныймыз, сонымен бірге осы қасиеттер қатарына оның вөлтерлігін де жатқызамыз.

**Түйін сөздер:** аргументі ауытқыған тендеу, толымдылық, базистік, вөлтерлік, Гаалдың формуласы, Лидскийдің теоремасы, Штурм –Лиувиллдің операторы, Рисстің базисі.

УДК 517.929

А.Ш.Шалданбаев<sup>1</sup>, С.М.Шаленова<sup>2</sup>, М.Б.Иванова<sup>3</sup>, А.А.Шалданбаева<sup>4</sup>

<sup>1</sup>Международный университет Silkway, г. Шымкент, Казахстан;

<sup>2</sup>Каспийский государственный университет технологий и инжиниринга им. Ш.Есенова, г.Ақтау, Казахстан;

<sup>3</sup>Южно-Казахстанская медицинская академия, г. Шымкент, Казахстан;

<sup>4</sup>Региональный социально-инновационный университет, г. Шымкент, Казахстан.

#### О СПЕКТРАЛЬНЫХ СВОЙСТВАХ КРАЕВОЙ ЗАДАЧИ УРАВНЕНИЯ ПЕРВОГО ПОРЯДКА С ОТКЛОНЯЮЩИМСЯ АРГУМЕНТОМ

**Аннотация.** В данной работе изучены спектральные свойства краевой задачи дифференциального уравнения первого порядка с постоянными коэффициентами и отклоняющимся аргументом, при этом отклонение присутствует при старшем члене уравнения, и его нельзя перебрасывать без дополнительного условия на младшие члены уравнения. Под спектральными свойствами мы имеем в виду полноту и базисность системы собственных и присоединенных функций краевой задачи, а также вольтерровость.

**Ключевые слова:** уравнение с отклоняющимся аргументом, полнота, базисность, вольтерровость, формула Гаала, теорема Лидского, оператор Штурма - Лиувилля, базис Рисса.

**Information about authors:**

Shaldanbayev A.Sh. – doctor of physico-mathematical Sciences, associate Professor, head of the center for mathematical modeling, «Silkway» International University, Shymkent; <http://orcid.org/0000-0002-7577-8402>;  
 Shalenova S.M. - Yessenov University; Aktau; <https://orcid.org/0000-0002-1378-3310>;  
 Ivanova M.B. - South Kazakhstan medical Academy, Shymkent; <https://orcid.org/0000-0001-6548-0699>;  
 Shaldanbayeva A.A. - "Regional Social-Innovative University", Shymkent; <https://orcid.org/0000-0003-2667-3097>

**REFERENCES**

- [1] Babbage C. An essay towards the calculus of functions, Part II., Philos Trans. Roy. Soc. London.106. p :179-256. London, 1816
- [2] Przevorska- Rolewicz D. Equations with transformed argument an Algebraic approach. Warszawa. 1973. 354 p.
- [3] Wiener J. Generalized solutions of functional differential equations, Singapore, World Sci. p:160-215, Singapore, 1993.
- [4] Jack K. Hale Sjoerd M.,Verduyn Lunel, Introduction to Functional Differential Equations, Springer Science+Business Media, LLC, 1993.
- [5] Cabada A., Tojo F.A.F. Differential Equations with Involutions. N.Y.: Atlantis Press, 2015.
- [6] Busenberg S. Stability and Hopf Bifurcation for a Population Delay Model with Diffusion Effects//Journal of Differential Equations. 1996. 124, Article No 0003.-p. 80-107
- [7] Cushing J.M. Bifurcation of Periodic Oscillations Due to Delays in Single Species Growth Models// J. Math. Biology.- 1978. 6.p. 145-161
- [8] Cushing J.M. Bifurcation of Periodic Solutions of Integro-differential Systems with Applications to time Delay Models in Population Dynamics//SIAM J. Appl. Math. December, 1977. Vol.33, No4.
- [9] Watkins W., Modified Wiener equations. Int. J. Math. Math. Sci.Vol-27. p: 347-356. 2001.
- [10] Watkins W.T., Asymptotic properties of differential equations with involutions. Int. J. Pure Appl. Math. Vol 44. p: 485-492, 2008.
- [11] Burlutskaya M.Sh., Khromov A.P., Justification of Fourier method in mixed problems with involution. // Izv. Sarat. unty. New Ser. Maths. Mechanics. Informatics.-2011. vol.11, №4. p. 3-12.
- [12] Burlutskaya M.Sh., Khromov A.P., Fourier method in mixed problem for a first-order partial differential equation with involution // Zh. Computed. Math and math. Ph. 2011. vol.51, №12. p. 2233-2246.
- [13] Keldysh M.V., On completeness of eigenfunctions of some classes of non-self-adjoint linear operators // Success of Math Sciences. 1967. vol. 26, No. 4 (160). p. 15-41.
- [14] Il'in V. A., Existence of a reduced system of eigen and associated functions for  $X_a$  non-self-adjoint ordinary differential operator // Number theory, mathematical analysis and their applications, Trudy Math. Inst. Steklov. 1976. 142. p. 148–155.
- [15] Il'in V.A. Necessary and sufficient conditions for basis property and equal convergence with trigonometric series of spectral expansions, 1 // Differential Equations. 1980. vol. 16, № 5. p. 771-794.
- [16] Il'in V.A. Necessary and sufficient conditions for basis property and equal convergence with trigonometric series of spectral expansions, 2 // Differential Equations. 1980. vol. 16, № 6. p. 980-1009.
- [17] Il'in V.A. Necessary and sufficient conditions for basis property in  $L_2$  and equal convergence with trigonometric series of spectral expansions and expansions in exponential systems // DAN SSSR. 1983. vol. 273, No. 4. p. 784-793.
- [18] Il'in V.A. On unconditional basis property of systems of eigenfunctions and associated functions of a second-order differential operator on a closed interval // DAN SSSR. 1983. vol. 273, No. 5. p. 1048-1053.
- [19] Il'in V.A. On relationship between type of boundary conditions and properties of basis property and equal convergence with trigonometric series of expansions of a non-self-adjoint differential operator by root functions // Differential Equations. 1994. V. 30, № 9. p. 1516-1529.
- [20] Otelbaev M. Estimates for S-numbers and completeness condition for a system of root vectors of a non-self-adjoint Sturm-Liouville operator, Math. Notes, 1979, Vol. 25, No. 3, p. 409-418.
- [21] Shkalikov A.A. Boundary value problems for ordinary differential equations with a parameter in boundary conditions // Trudy Sem. by I.G.Petrovsky. 1983. Vol. 9. p. 190-229.
- [22] Gomilko A.M., Radzievsky G.V. Basis properties of eigenfunctions of a regular boundary value problem for a vector functional differential equation // Differential Equations. 1991. vol. 27, No.3. p. 385-395.
- [23] Il'in V.A., Kritskov L. V. Properties of spectral expansions corresponding to non-self-adjoint differential operators // J. Math. Sci. (NY). 2003. 116, N 5. P. 3489–3550.

[24] Il'in V.A., Kritskov L.V. Properties of spectral expansions corresponding to non-self-adjoint operators, Functional analysis. Results of science and technology. Ser. :. Rec.math. and its app. Them.view, M: VINITI, 2006, vol.96, p.5-105.

[25] Ibraimkulov A.M. On spectral properties of boundary value problem for an equation with deviating argument. News of AS KazSSR, ser. of ph. math. 1988. №3. p.22-25.

[26] Shaldanbayev A.Sh., Akhmetova S.T., On completeness of eigenvectors of the Cauchy problem. // Republican scientific journal "Science and Education of the SK" No.27, 2002. p. 58-62.

[27] Akhmetova S.T., On Bitsadze-Samarskii Problem for the Wave Equation, Mathematical Journal, Almaty. 2003, Vol. 3. №2 (8), p.15-18.

[28] Kalmenov T.Sh., Akhmetova S.T., Shaldanbayev A.Sh. To spectral theory of equations with deviating argument, Mathematical Journal, Almaty.2004.vol. 4, No. 3. P. 41-48.

[29] Shaldanbaev A.Sh. Criteria for Volterra property of differential operator of the first order with deviating argument, Bulletin of Karaganda University, "Mathematics" series, № 3 (47) / 2007, p.39-43.

[30] Sadybekov M.A., Sarsenbi A.M. Unconditional convergence of spectral expansions associated with a second-order differential equation with deviating argument. // Bulletin of KazNU. 2006. №2 (49) . p.48-54.

[31] Sadybekov M.A., Sarsenbi A.M. Solution of main spectral problems of all boundary value problems for a single differential equation of the first order with deviating argument. // Uzbek Mathematical Journal. 2007. №3. p.88-94.

[32] Sadybekov M.A., Sarsenbi A.M. On notion of regularity of boundary value problems for a second order differential equation with deviating argument. // Mathematical journal. 2007. vol.7, №1 (23). p. 82-88

[33] Sadybekov M.A., Sarsenbi A.M. Criterion for basis property of a system of the eigenfunctions of a multiple differentiation operator with involution. // Differential equations - 2012. vol.48, No. 8. p. 1126-1132

[34] Kopzhasarova A.A, Sarsenbi A.M. Basis Properties of Eigenfunctions of Second-Order Differential Operators with Involution//Abstract and Applied Analysis. 2012. Volume 2012, Article ID 576843. 6 pages doi:10.1155/2012/576843.

[35] Kopzhasarova A. A., Lukashov A. L., Sarsenbi A. M. Spectral properties of non-self-adjoint perturbations for a spectral problem with involution // Abstr. Appl. Anal. 2012. p. 1–5.

[36] Sarsenbi A.M., Tengaeva A.A. On basis properties of root functions of two generalized eigenvalue problems// Differential Equations.-2012.- vol. 48, no. 2.- p. 306–308

[37] Imanbaeva A.B., Kopzhasarova A.A., Shaldanbaev A.Sh., Asymptotic expansion of solution of a singularly perturbed Cauchy problem for a system of ordinary differential equations with constant coefficients.\\ "News of NAS RK. Physics and Mathematics Series, 2017, No. 5, 112-127.

[38] Kopzhasarova A.A., Shaldanbaev A.Sh., Imanbaeva A.B. Solution of singularly perturbed Cauchy problem by the similarity method. "News of NAS RK. Physics and Mathematics Series, 2017, no 5, p.127-134.

[39] Amir Sh. Shaldanbayev, Manat T. Shomanbayeva ,Solution of singularly perturbed Cauchy problem for ordinary differential equation of second order with constant coefficients by Fourier method ,Citation: AIP Conference Proceedings 1880, 040017 (2017); doi: 10.1063/1.5000633 View online: <http://dx.doi.org/10.1063/1.5000633> View Table of Contents: <http://aip.scitation.org/toc/apc/1880/1> Published by the American Institute of Physics.

[40] Akulbaev M.I., Beisebaeva A., Shaldanbaev A. Sh. On periodic solution of the Goursat problem for a wave equation of a special form with variable coefficients (in English), News of NAS RK. Physical and mathematical Series. № 1. 2018, p.34-50.

[41] Shaldanbaeva A. A., Akylbayev M.I., Shaldanbaev A. Sh., Beisebaeva A.Zh., The spectral decomposition of cauchy problem's solution for laplace equation, News of the National Academy of Sciences of the Republic of Kazakhstan Physico-mathematical Series, Issn 1991-346X. <https://doi.org/10.32014/2018.2518-1726.10> Volume 5, Number 321 (2018), 75–87.

[42] Shaldanbayev A.Sh., Shaldanbayeva A.A., Shaldanbay B.A., On projectional orthogonal basis of a linear non-self -adjoint operator, News of the national academy of sciences of the republic of Kazakhstan Physico-mathematical series7,Issn 1991-346X <https://doi.org/10.32014/2019.2518-1726.15> Volume 2, Number 324 (2019), 79 – 89.

[43] Akylbayev M.I., Shaldanbayev A.Sh., Orazov I., Beysebayeva A.. About single operator method of solution of a singularly perturbed Cauchy problem for an ordinary differential equation  $n$  – order, News of the national academy of sciences of the republic of Kazakhstan Physical-mathematical series, Issn 1991-346X <https://doi.org/10.32014/2019.2518-1726.8>,Volume 2, Number 324 (2019), 17 – 36.

[44] Ashyralyev A., Sarsenbi A. M. Well-posedness of an elliptic equations with an involution // EJDE. 2015. 284. P. 1–8.

[45] Kritskov L. V., Sarsenbi A. M. Spectral properties of a nonlocal problem for the differential equation with involution // Differ. Equ. 2015. 51, N 8. P. 984– 990.

[46] Kurdyumov V. P. On Riesz bases of eigenfunction of 2-nd order differential operator with involution and integral boundary conditions // Izv. Saratov Univ. (N.S.), Ser. Math. Mech. Inform. 2015. 15, N 4. P. 392–405.



- [47] Kirane M., Al-Salti N. Inverse problems for a nonlocal wave equation with an involution perturbation // *J. Nonlinear Sci. Appl.* 2016. 9. P. 1243–1251.
- [48] Baranetskij Ya. O., Kolyasa L. I., Boundary value problem for second order differential operator equation with involution, *Vestnik Nath. University, "Lvovskaya polytech", "Phys.-math.science"*, № 871, 2017, p. 20–26.
- [49] Kritskov L.V., Sarsenbi A.M., Equiconvergence property for spectral expansions related to perturbations of the operator  $u''(x)$  with initial data, *Filomat*, 2018, том 32, № 3, p. 1069-1078.
- [50] Kritskov L.V., Sadybekov M.A., Sarsenbi A.M., Nonlocal spectral problem for a second-order differential operator with an involution // *Bulletin of Karaganda University -MATHEMATICS*, 2018 № 3 (91), p. 53-60.
- [51] Kritskov L.V., Sadybekov M.A., Sarsenbi A.M., Properties in  $L_p$  of root functions for a nonlocal problem with involution, *Turkish Journal of Mathematics*, Scientific and Technical research Council of Turkey - TUBITAK/Turkiye Bilimsel ve Teknik Arastirma Kurumu (Turkey), 2019, vol. 43, p. 393-401.
- [52] Vladykina V.E., Shkalikov A.A., Spectral Properties of Ordinary Differential Operators with Involution, *Dokl. Akad. Nauk*, 2019, Vol. 484, No. 1, p. 12–17.
- [53] Shaldanbaev A.Sh., Kulazhanov G.S. On spectral properties of one model operator with deviating argument // *International Conference VEKUA-100, Novosibirsk 2007*.
- [54] Bari N.K. Biorthogonal systems and bases in the Hilbert space, // *Uch.zap.MGU. 1951. vol.4, Issue.148. p. 69-106*.
- [55] Gokhberg I.T., Krein M.G. Introduction to the theory of linear non-self-adjoint operators in a Hilbert space, M: Nauka, 1965, 448 p.
- [56] Naimark M.A. *Linear Differential Operators*. M.: Science, 1969. 528 p.
- [57] Kesselman G.M. On unconditional convergence of expansions in eigen functions of some differential operators. // *News of univ. Math.* 1964. № 2 (39). p. 82-93.
- [58] Mikhailov V.P. On Riesz bases in  $L_2(0,1)$ . // *DAN USSR*. 1962. vol. 144, № 5. p. 981-984.
- [59] Lidsky V.B. Non-self-adjoint operators having a trace. *DAN USSR*, 1959, vol.125. No.3, p. 485-488.
- [60] Brislawn C., Kernels of trace class operators. // *American Mathematical Society*, V.104, № 4, 1988, p. 1181-1190.

**Publication Ethics and Publication Malpractice  
in the journals of the National Academy of Sciences of the Republic of Kazakhstan**

For information on Ethics in publishing and Ethical guidelines for journal publication see <http://www.elsevier.com/publishingethics> and <http://www.elsevier.com/journal-authors/ethics>.

Submission of an article to the National Academy of Sciences of the Republic of Kazakhstan implies that the described work has not been published previously (except in the form of an abstract or as part of a published lecture or academic thesis or as an electronic preprint, see <http://www.elsevier.com/postingpolicy>), that it is not under consideration for publication elsewhere, that its publication is approved by all authors and tacitly or explicitly by the responsible authorities where the work was carried out, and that, if accepted, it will not be published elsewhere in the same form, in English or in any other language, including electronically without the written consent of the copyright-holder. In particular, translations into English of papers already published in another language are not accepted.

No other forms of scientific misconduct are allowed, such as plagiarism, falsification, fraudulent data, incorrect interpretation of other works, incorrect citations, etc. The National Academy of Sciences of the Republic of Kazakhstan follows the Code of Conduct of the Committee on Publication Ethics (COPE), and follows the COPE Flowcharts for Resolving Cases of Suspected Misconduct ([http://publicationethics.org/files/u2/New\\_Code.pdf](http://publicationethics.org/files/u2/New_Code.pdf)). To verify originality, your article may be checked by the Cross Check originality detection service <http://www.elsevier.com/editors/plagdetect>.

The authors are obliged to participate in peer review process and be ready to provide corrections, clarifications, retractions and apologies when needed. All authors of a paper should have significantly contributed to the research.

The reviewers should provide objective judgments and should point out relevant published works which are not yet cited. Reviewed articles should be treated confidentially. The reviewers will be chosen in such a way that there is no conflict of interests with respect to the research, the authors and/or the research funders.

The editors have complete responsibility and authority to reject or accept a paper, and they will only accept a paper when reasonably certain. They will preserve anonymity of reviewers and promote publication of corrections, clarifications, retractions and apologies when needed. The acceptance of a paper automatically implies the copyright transfer to the National Academy of Sciences of the Republic of Kazakhstan.

The Editorial Board of the National Academy of Sciences of the Republic of Kazakhstan will monitor and safeguard publishing ethics.

Правила оформления статьи для публикации в журнале смотреть на сайтах:

[www:nauka-nanrk.kz](http://www.nauka-nanrk.kz)

<http://physics-mathematics.kz/index.php/en/archive>

**ISSN 2518-1726 (Online), ISSN 1991-346X (Print)**

Редакторы *М. С. Ахметова, Т.А. Апендиев, Д.С. Аленов*  
Верстка на компьютере *А.М. Кульгинбаевой*

Подписано в печать 10.10.2019.  
Формат 60x881/8. Бумага офсетная. Печать – ризограф.  
9,6 п.л. Тираж 300. Заказ 5.

---

*Национальная академия наук РК*  
*050010, Алматы, ул. Шевченко, 28, т. 272-13-18, 272-13-19*